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OPTIMAL CONTROL IN EVOLUTIONARY MICROMAGNETISM
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Abstract. We consider an optimal control problem subject to the Landau-Lifshitz-Gilbert
equation which describes the evolution of magnetizations in $S^2$. The problem is motivated in
order to control switching processes of ferromagnets. Existence of an optimum and the first
order necessary optimality system are derived. We show (up to subsequences) convergence
of state, adjoint and control variables of a time discretization (semi-implicit Euler method)
for vanishing time step size. A main step here is to verify corresponding stability properties
for the semi-discrete state, which is nontrivial since the iterates take values which only
approximate $S^2$. We use a perturbation argument within a variational discretization in order
to show error bounds for the semi-discrete state variables, from which we may then infer
uniform bounds for the semi-discrete state and also adjoint variables. Numerical studies
underline these results and compare this discretization with a further variant which bases on
a projection strategy for the state equation to enhance iterates to better approximate $S^2$.

1. Introduction

The aim of this work is to control the magnetization inside a ferromagnetic material by means
of an external field $H_{\text{eff}} : (0, T) \times \Omega \to \mathbb{R}^3$. The control problem is motivated to e.g. optimize
the switching dynamics in data storage media \cite{[8],[15]}, where external forces $u : (0, T) \times \Omega \to \mathbb{R}^3$
are responsible for writing and reading phenomena.

The main problem of this work reads as follows.

Problem 1.1. Let $T > 0$, $\alpha > 0$, $\Omega \subset \mathbb{R}^d$ (for $d \geq 1$) be bounded and Lipschitz, and $m_0 : \Omega \to \mathbb{R}^3$, and $\tilde{m} : [0, T] \times \Omega \to \mathbb{R}^3$, respectively, be given functions. Find $m^* : [0, T] \times \Omega \to \mathbb{R}^3$
and $u^* : (0, T) \times \Omega \to \mathbb{R}^3$, such that

$$(m^*, u^*) = \arg\min_{m, u} \frac{1}{2} \int_0^T \int_{\Omega} |m - \tilde{m}|^2 \, dx \, dt + \frac{\lambda}{2} \int_0^T \int_{\Omega} |u|^2 \, dx \, dt$$

subject to the Landau-Lifshitz-Gilbert (LLG) equation,

$$m_t = -\alpha m \times (m \times H_{\text{eff}}(m)) + m \times H_{\text{eff}}(m) \quad \text{in } (0, T) \times \Omega,$$

(1.1)

together with the initial condition $m(0) = m_0$, and suitable boundary conditions.

The Landau-Lifshitz-Gilbert equation describes the evolution of the magnetization $m : [0, T] \times 
\Omega \to \mathbb{R}^3$ in the presence of a given field $H_{\text{eff}} : (0, T) \times \Omega \to \mathbb{R}^3$. The effective magnetic

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field $H_{\text{eff}}(m)$ is given by the negative derivative of the physical energy $E(m)$ related to the magnetization progress; it includes the exchange energy, the magneto-crystalline anisotropy energy, the (nonlocal) magnetostatic energy, as well as an external field energy modelled by the force $u : (0, T) \times \Omega \to \mathbb{R}^3$. In this work, we only take the exchange energy and the external field into account, i.e., we set $H_{\text{eff}}(m) = -D\dot{E}(m) = \Delta m + u$. We refer to [14] and the references therein for a recent summary about the model, the physical meaning of the energies, and known results concerning the equation (1.1).

The Landau-Lifshitz-Gilbert equation has three properties which are essential for the following: Firstly, elementary properties of the cross-product yield to the pointwise identity $\frac{d}{dt}|m(t,)|^2 = 0$ in (1.1), i.e., $m$ and $m_t$ are pointwise orthogonal in space and time, see Figure 1(a). Also, this yields $|m|^2 = 1$ pointwise in time and space for $m_0(.) \in S^2$. Secondly, the pointwise multiplication of the equation (1.1) with $-\Delta m$ leads to the energy inequality

$$\frac{1}{2} \|\nabla m(t)\|^2_{L^2(\Omega)} \leq C \int_0^t \|u\|^2_{L^2(\Omega)} \, ds + \frac{1}{2} \|\nabla m_0\|^2_{L^2(\Omega)}. \quad (1.2)$$

From a geometric viewpoint, the dissipative term $-\alpha m \times (m \times \Delta m)$ in (1.1) may be interpreted as the projection of $\Delta m$ onto the two-dimensional sphere $S^2 \subset \mathbb{R}^3$. As a consequence, we have the identity

$$-m \times (m \times \Delta m) = |\nabla m|^2 m + \Delta m, \quad (1.3)$$

where the factor $|\nabla m|^2$ in the first term on the right hand side may be seen as a Lagrange multiplier associated to the constraint that $m$ is $S^2$-valued. The identity (1.3) then leads to the following reformulation of (1.1),

$$m_t - \alpha \Delta m = \alpha |\nabla m|^2 m - \alpha m \times (m \times u) + m \times (\Delta m + u). \quad (1.4)$$

The formulation (1.4) illustrates the semilinear character of the PDE and helps in combination with the energy inequality (1.2) to show strong regularity results for weak solutions of (1.4); cf. Lemma 3.3. Moreover, the formulation (1.4) will be the starting point for the optimization problem – in particular for the derivation of the necessary optimality system.

In this work, we discuss solvability of Problem 1.1 and derive the first order necessary optimality system for a minimizer by means of a practical time discretization scheme for (1.4). We use the Lagrange multiplier theorem for this purpose, and benefit from improved regularity properties for solutions of the equation (1.4) for $d = 1$. A main ingredient for our overall study are known results on the analysis and numerics of (1.1), where $u = 0$ in $(0, T) \times \Omega$; cf. [11, 18].

The choice of a discretization scheme is not immediate since it is desirable to preserve the properties of the continuous state equation. A key property to study stability and convergence of the optimality system of the discrete version of Problem 1.1 are uniform estimates in strong norms for the iterates. We discuss different strategies for semi-discrete schemes which highlight those problems:

1. A scheme which bases on the use of the midpoint rule for equation (1.1), (cf. [3, 6]) governs a discrete dynamics on $S^2$ and fulfills a discrete energy equality. The scheme uses the quantity $m^{j+\frac{1}{2}} := \frac{1}{2}(m^j + m^{j+1})$. A problem occurs in the derivation of $H^2(\Omega)$-stability of each iterate. For this purpose, we want to reformulate the scheme by using (1.3), where the leading terms in the reformulated scheme now read as
\[ d_t \mathbf{m}^{j+1} - \alpha |\mathbf{m}^{j+1/2}|^2 \Delta \mathbf{m}^{j+1} \] As a consequence, the reformulation may be degenerate elliptic since only \(|\mathbf{m}^{j+1/2}| \geq 0\), and hence uniform bounds for iterates \(\{\mathbf{m}^{j+1}\}\) in \(H^2(\Omega)\) are not immediate. This missing property later causes problems in the stability analysis for the adjoint equation (where solutions of the discrete state equation are coefficients), and hence convergence of a corresponding scheme to the optimality system for Problem 1.1 remains unclear.

We also mention here the time-explicit structure respecting discretizations [2, 4] which bases on the tangent plane reformulation of (1.1); as a consequence, the numerical treatment of the related Problem 1.1 would require a different approach to the one presented below.

(2) Iterates of the discretization by the semi-implicit Euler method (1.5) for (1.4) are analyzed in Section 5. For regular solutions of (1.4), iterates of (1.5) may be shown to satisfy a discrete energy estimate which corresponds to (1.2), and are uniformly bounded in even stronger norms. This result is shown by a perturbation argument and thus hinges on existing strong solutions for (1.4) which are known to exist for \(\Omega \subset \mathbb{R}^3\), and sufficiently regular data \(\mathbf{m}_0\) and \(\mathbf{u}\). It is in contrast to the continuous counterpart that an energy (in-)equality is not obtained via the formulation (1.1) since iterates of this method are in general not \(S^2\)-valued (a typical situation is illustrated in Figure 1(b)). We remark that to use a fully implicit Euler method leaves unclear related stability and convergence properties, while the semi-implicit discretization (1.5) below allows for an inductive argumentation to establish those properties.

(3) In order to compensate for the drawback of the semi-implicit Euler method (1.5) to only generate \(\mathbb{R}^3\)-valued iterates, a strategy is to modify the semi-implicit Euler scheme by projecting some of the old time step iterates onto \(S^2\). For \(\mathbf{u} \equiv 0\), it is known [18] that such a scheme inherits stability properties of the original scheme (1.5), and the dynamics is now much closer to \(S^2\). In fact, the additional projection step may be re-interpreted by means of an additional penalty term \(\frac{1}{k} (1 - \frac{1}{|\mathbf{m}|}) \mathbf{m}^j\) on the left hand side of (1.5). In Section 8 we formally derive the corresponding discrete optimality system for this method, but we have to leave open a corresponding convergence analysis and discuss new theoretical problems which arise with this method. In particular, the adjoint equation is now enriched by further terms which eventually go back to the pointwise projection.

It is known that strong stability properties of the iterates are relevant in order to show stability of the adjoint equation, and then to pass to the limit in both, the state and adjoint equation. As a consequence, we give up the requirement for iterates to take values in \(S^2\) and use the second approach for a time discretization for the Landau-Lifshitz-Gilbert equation. The scheme reads as follows.

Scheme 1.2. Let \(\mathbf{m}^0 := \mathbf{m}_0\). For every \(j \geq 0\) let \(\mathbf{m}^{j+1} : \Omega \to \mathbb{R}^3\) be the solution of

\[ \frac{1}{k} (\mathbf{m}^{j+1} - \mathbf{m}^j) - \alpha \Delta \mathbf{m}^{j+1} = \alpha |\nabla \mathbf{m}^j|^2 \mathbf{m}^{j+1} - \alpha \mathbf{m}^{j+1} \times (\mathbf{m}^j \times \mathbf{u}^{j+1}) + \mathbf{m}^{j+1} \times \Delta \mathbf{m}^{j+1} + \mathbf{m}^{j+1} \times \mathbf{u}^{j+1}. \] (1.5)

Optimal rates of convergence at finite times for iterates of (1.5) towards regular solutions of (1.4) have been shown in [18] by means of an inductive argument which compensates for the lack for the discrete version of the energy inequality (1.2). This argument uses strong norms
of solutions of \((1.4)\), in particular uniform bounds for second time derivatives of the solution, which are not available in Problem \(1.1\) due to the limited (temporal) regularity of the control \(u\). A first step in this work is therefore to sharpen this induction argument by reducing regularity requirements on solutions \((m, u)\) in Problem \(1.1\) (for \(d = 1\)): In order to do so, we use a variational argument from [19] to verify bounds for iterates \(\{m_j\}\) of \((1.5)\) taking values in \(H^2(\Omega)\). The bounds will depend on the regularity of the solution \(m\) of \((1.4)\), which is again limited by the one of \(u\). These strong stability properties for the continuous problem \((1.4)\) and the discrete scheme \((1.5)\), respectively, are the key ingredients to show convergence of the discrete optimality system which is based on \((1.5)\) towards the corresponding necessary optimality system for Problem \(1.1\).

The formal derivation of the necessary optimality condition for minima \((m^*, u^*)\) of Problem \(1.1\) yields the following geometric property:
\[
\lambda u^* = (-\alpha (z \times m^*) + z) \times m^*,
\]
where \(z\) is the associated adjoint variable solving the adjoint equation \((4.1c)\). This equation implies that the optimal control \(u^*\) is orthogonal to the optimal state \(m^*\), i.e., for all \((t, x) \in (0, T) \times \Omega\) we have \(u^*(t, x) \in T_xS^2\), see Figure 1(c). Correspondingly, the necessary optimality condition for a minimum \((\{m_j^*\}, \{u_j^*\})\) of the discrete optimization Problem 6.1 reads
\[
\lambda u_j^* = -\alpha (z_j^{-1} \times m_j^*) \times m_j^{i-1} - z_j^{-1} \times m_j^i,
\]
\((1.7)\), together with \((1.5)\), where \(z_j^{i-1}\) is the associated adjoint variable solving \((6.1c)\). The condition \((1.7)\) is illustrated in Figure 1(d). We observe that in the semi-discrete setting, the shifted iterates \(m_j^{i-1}\) or \(m_j^i\) and \(u_j^i\) are nearly orthogonal. If we use the projection method which is mentioned in [3] as the state equation we get the same optimality condition. In this case, numerical experiments indicate an additional (approximate) orthogonality between the optimal state \(m_j^i\) and the adjoint \(z_j^i\); see Section 8.3.2.

So far, we cannot analyze Problem \(1.1\) in the present form because of some severe technical issues which we discuss now. These issues lead to some modifications of Problem \(1.1\)
(1) It is due to the strong nonlinearities in (1.1) that possible finite time blow-up of regular solutions may be expected for domains \( \Omega \subset \mathbb{R}^d \) for \( d \geq 2 \), cf. [11]. As a consequence, we restrict our analysis to the one-dimensional case \( (d = 1) \). In order to suppress boundary effects, we consider \( \Omega := S^1 \).

(2) When using the perturbation argument to obtain strong stability properties for the solution of the semi-discrete equation (1.5), it turns out that we need additional regularity properties for the continuous solution of (1.1). The bottleneck for the regularity of \( \mathbf{m} \) is the regularity given for the external force \( \mathbf{u} \) in (1.1). We achieve our overall goal of the work by replacing the functional in Problem 1.1 by

\[
F(\mathbf{m}, \mathbf{u}) = \frac{1}{2} \int_0^T \int_{\Omega} |\mathbf{m} - \tilde{\mathbf{m}}|^2 \, dx \, dt + \frac{\lambda_0}{2} \int_0^T \int_{\Omega} |\mathbf{u}|^2 \, dx \, dt + \frac{\lambda_1}{2} \int_0^T \int_{\Omega} |\nabla \mathbf{u}|^2 \, dx \, dt
\]

for \( \lambda_0, \lambda_1 > 0 \). Then every feasible point of the modified optimization problem has the desired regularity. This modification will also effect the pointwise orthogonality relation (1.6), which is transformed into an abstract orthogonality relation on Sobolev spaces.

To derive the optimality system of the semi-discrete optimization problem we follow the strategy ‘first discretize, then optimize’, i.e., we first set up the semi-discrete optimization problem before we derive the necessary optimality system. The advantage of this ansatz is that we may immediately infer solvability of the adjoint equation by the Lagrange multiplier theorem, and this approach leads to a natural semi-discretization of the adjoint equation as well. Moreover, the regularity properties of the state and the adjoint variables are similar (cf. Lemmas 3.3 and 4.7).

To our knowledge, the first studies of the necessary optimization system for optimal control type problems with the Landau-Lifshitz-Gilbert equation as the state equation are done in the recent works [1] and [3]. In these works, solvability is shown and the necessary optimality system is derived. The results are obtained for magnetizations which are constant in space, which leads to an optimization problem subject to an ordinary differential equation. It seems that our work is the first which deals with the PDE (1.1).

Optimization subject to the linear heat equation with a general discontinuous Galerkin scheme was studied in [17]. There, also \( H^2(\Omega) \)-bounds for iterates are needed to prove an optimal convergence rate – while we need corresponding bounds in order to control nonlinearities in the discrete scheme, and we only show convergence in the adjoint equation without rates.

The paper is organized as follows. In Section 3, we consider the non-homogeneous Landau-Lifshitz-Gilbert equation, and recall well-known solvability and regularity properties. In Section 4, we state the continuous optimization Problem 4.1 where all the changes indicated above are accounted for, and prove solvability; see Theorem 4.2. Finally, we derive the necessary optimality system (4.1) with the help of the Lagrange multiplier theorem.

Starting with Section 5, we introduce the semi-discrete equation (5.1), show solvability of this semi-implicit scheme for sufficiently small time steps \( k > 0 \) and verify strong stability results in Theorem 5.4 and Lemma 5.8. In Section 6, we establish solvability for the semi-discrete optimization Problem 6.1 in Theorem 6.2. The derivation of the semi-discrete optimality system (6.1) will be done by the Lagrange multiplier theorem again, leading to a practically relevant construction of a solution of the optimization Problem 4.1. The main result (Theorem 7.4)
is then studied in Section 8.3.2. Up to a subsequence, the optimal state, adjoint and control variables of the semi-discrete optimization problem converge to a solution of the continuous optimality system (4.1). In addition, we show that control variables converge even strongly in $L^2(H^1)$ to a continuous control (up to a subsequence).

In Section 8 we present some computational studies: First we introduce the projection Scheme 8.1 and set up the necessary optimality system corresponding to Problem 4.1. Then we compare results of the semi-implicit Euler method (5.1) and the projection scheme for different choices of the constant $\lambda_0$ in the functional; cf. Sections 8.3 and 8.4. There, we will also see results concerning orthogonalities between the optimal state, control and adjoint; cf. Section 8.3.2.

2. Preliminaries

Let $L^p(\Omega)$ and $H^k(\Omega) := W^{k,2}(\Omega)$ denote standard Lebesgue and Sobolev spaces. For $T > 0$ let $H^k(H^m) := H^k(0,T; H^m(\Omega))$ denote a standard Bochner space, see e.g. [22]. Vector-valued functions, and spaces containing such functions are written in bold-face notation. For the scalar product in $L^2$ of $f, g : \Omega \to \mathbb{R}$ we write $(f,g)$, and the inner product of a Banach space $X$ and its dual space $X^*$ is written as $(\cdot, \cdot)$.

We denote by $C$ and $K$ generic nonnegative constants; to indicate dependencies, we e.g. write $C(.)$.

**Semi-discrete setup.** For $k = \frac{T}{J}$, let $I_k = \{t_j\}_{j=0}^J$ be an equi-distant partition of $[0, T]$ with mesh size $k > 0$. Let $\{v^j\}_{j=0}^J \subset X$ be a given sequence. For every fixed $0 \leq j < J$ we denote by $d_t v^{j+1} := \frac{v^{j+1} - v^j}{k}$ the discrete time derivative. The piecewise affine, globally continuous time interpolant $\mathcal{V} \in \mathcal{C}(X)$ of $\{v^j\}_{j=0}^J$ is defined via

$$\mathcal{V}(t) := \frac{t-t_j}{k} v^{j+1} + \frac{t_{j+1}-t}{k} v^j$$

for $t \in (t_j, t_{j+1}]$, and the following piecewise constant in time interpolants $\mathcal{V}^+, \mathcal{V}^-$ and $\mathcal{V}^*$ of $\{v^j\}_{j=0}^J$ satisfy

$$\mathcal{V}^+(t) := v^{j+1}, \quad \mathcal{V}^-(t) := v^j, \quad \mathcal{V}^*(t) := v^{j+2}$$

for $t \in (t_j, t_{j+1}]$, with $v^1 := v^0$ and $v^{J+1} := 0$. Correspondingly, we define for $v \in \mathcal{C}(L^2)$ the piecewise constant in time interpolants $v^+$ and $v^-$ of $v$,

$$v^+(t) := v(t_{j+1}), \quad v^-(t) := v(t_j)$$

for $t \in (t_j, t_{j+1}]$.

3. The state equation with external force

In this section we will consider the state equation (LLG). We recall the notion of a strong solution and related regularity properties in the presence of applied regular forces.

Given an applied force $u : [0,T] \times \Omega \to \mathbb{R}^3$, the magnetization $m : [0,T] \times \Omega \to \mathbb{R}^3$ fulfills the following equation (LLG)

$$m_t = -\alpha m \times (m \times (\Delta m + u)) + m \times (\Delta m + u) \quad \text{on } (0,T) \times \Omega,$$

(3.1)

together with the initial condition $m(0,\cdot) = m_0$ on $\Omega$. 

Definition 3.1 (Strong solution). Suppose \( u \in L^2(L^2) \). We call \( m : [0, T] \times \Omega \to \mathbb{R}^3 \) a strong solution of (3.1) if

1. \( m \in H^1(L^2) \cap L^2(H^2) \to C(H^1), \)
2. for almost all \( t > 0 \) and for all \( \varphi \in C^\infty(\Omega) \) holds
   \[
   (m_t, \varphi) = \alpha (\nabla m \times (m \times \nabla m), \varphi) + \alpha (m \times (m \times \nabla m), \nabla \varphi) - \alpha (m \times (m \times u), \varphi) - (m \times \nabla m, \nabla \varphi) + (m \times u, \varphi),
   \]
3. for all \( (t, x) \in [0, T] \times \Omega \) holds \( |m| = 1 \),
4. for all \( t \in [0, T] \) the energy equality is valid
   \[
   \frac{1}{2} \|\nabla m(t)\|_{L^2(\Omega)}^2 + \alpha \int_0^t \|m \times \Delta m\|^2_{L^2(\Omega)} \, ds
   = \int_0^t (m \times \Delta m, u - \alpha m \times u) \, ds + \frac{1}{2} \|\nabla m_0\|_{L^2(\Omega)}^2,
   \]
5. \( m \) fulfills the initial condition \( m(0) = m_0 \in H^1(\Omega) \).

An important observation for the following is an equivalent reformulation of (3.1): Given an applied force \( u : (0, T) \times \Omega \to \mathbb{R}^3 \), the magnetization \( m : [0, T] \times \Omega \to \mathbb{R}^3 \) fulfills

\[
m_t - \alpha \Delta m = \alpha (\nabla m \times m \times (m \times u) + m \times (\Delta m + u)) \quad \text{on} \ (0, T) \times \Omega, \tag{3.2}
\]

together with the initial condition \( m(0, .) = m_0 \) on \( \Omega \); here the notion of a strong solution is analogous to Definition 3.1.

The following assertion shows the equivalence of the two formulations above, cf. [11, Theorem 1.1].

Lemma 3.2. Let \( u \in L^2(L^2), m_0 \in H^1(\Omega) \) with \( |m_0|^2 = 1 \) everywhere in \( \Omega \). Then \( m : [0, T] \times \Omega \to \mathbb{R}^3 \) is a strong solution of (3.1) if and only if it is a strong solution of (3.2).

We recall the following well-known results for the LLG equation, see [11].

Lemma 3.3. Let \( T > 0, u \in L^2(L^2), \) and \( m_0 \in H^1(\Omega) \) with \( |m_0|^2 = 1 \) everywhere in \( \Omega \). There exists a unique strong solution \( m : [0, T] \times \Omega \to \mathbb{R}^3 \) of (3.1). Moreover, the following estimates are valid,

1. \( \|m\|^2_{L^\infty(H^1)} + \|m \times \Delta m\|^2_{L^2(L^2)} + \|m\|^2_{H^1(L^2)} \leq C \left( \|u\|^2_{L^2(L^2)} + \|m_0\|^2_{H^1(\Omega)} \right), \)
2. \( \|m\|^2_{L^2(H^2)} \leq C \left( \|u\|_{L^2(L^2)} + \|m_0\|_{H^1(\Omega)} \right), \)

Proof. To verify (2), formally multiply (3.2) with \( -\Delta m \) and interpolate \( L^4(\Omega) \) between \( L^2(\Omega) \) and \( H^1(\Omega) \).

We remark that assertion (2) in Lemma 3.3 is obtained via an interpolation estimate for Sobolev norms, which in this form is not valid any more for a two-dimensional domain \( \Omega \subset \mathbb{R}^2 \).

For the stability proof for the semi-discretization (5.1) below (see Theorem 5.4), we have to employ improved regularity properties of the strong solution \( m \) of (3.1) in the presence of more regular controls \( u \in L^2(H^1) \). For example it will be necessary to guarantee \( m \in H^1(H^1) \).
Let $T > 0$, $u \in L^2(H^1)$, and $m_0 \in H^2(\Omega)$ with $|m_0|^2 = 1$ everywhere in $\Omega$. Let $m$ be the strong solution of (3.1). There exists a constant $C = C(\|u\|_{L^2(H^1)} \cdot \|m_0\|_{H^2(\Omega)}, T)$ such that

$$\|m\|^2_{L^\infty(H^1)} + \|m\|^2_{L^2(H^2)} + \|m\|^2_{H^1(H^1)} \leq C.$$ 

**Proof.** Differentiate (3.2) formally in space. With $A := \nabla m$ we get

$$A_t - \alpha \Delta A = 2\alpha(\nabla A, \nabla m)m + \alpha |\nabla m|^2 A - \alpha (A \times (m \times u)) - \alpha (m \times (A \times u)) - \alpha (m \times (m \times \nabla u)) + A \times \Delta m + m \times \Delta A + A \times u + m \times \nabla u. \quad (3.3)$$

Multiply (3.3) with $-\Delta A$ and $A_t$ and integrate in space and time. Use $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ and a Gagliardo-Nirenberg inequality for $L^4(\Omega)$ as above to bound arising terms. \qed

4. Analysis of the optimization problem

Next, we formulate the optimization problem with the LLG equation (3.2) as constraint and deduce the necessary optimality system for a minimum with the help of the Lagrange multiplier theorem. Define the spaces

$$M := L^2(H^2) \cap H^1(L^2) \hookrightarrow C(H^1) \hookrightarrow L^\infty(L^\infty) \quad \text{and} \quad U := L^2(H^1),$$

as well as for some given $\bar{m} \in C^\infty(C^\infty)$ and $\lambda_0, \lambda_1 > 0$ the functional

$$F : M \times U \rightarrow \mathbb{R}, \quad F(M, U) := \frac{1}{2} \|m - \bar{m}\|^2_{L^2(L^2)} + \frac{\lambda_0}{2} \|u\|^2_{L^2(L^2)} + \frac{\lambda_1}{2} \|
abla u\|^2_{L^2(L^2)}.$$ 

Now we can describe the optimization problem with a desired state $\bar{m} : [0, T] \times \Omega \rightarrow \mathbb{S}^2$.

**Problem 4.1.** Let $\bar{m} : [0, T] \times \Omega \rightarrow \mathbb{S}^2$ be smooth and given, as well as $m_0 \in H^2(\Omega)$ with $|m_0|^2 = 1$ in $\Omega$, and $\lambda_0, \lambda_1 > 0$. Find $m^* \in M$ and $u^* \in U$, such that

$$(m^*, u^*) = \arg\min_{m \in M, u \in U} F(m, u) \quad \text{subject to} \quad (3.2).$$

4.1. Existence.

**Theorem 4.2.** There exists at least one minimum $(m^*, u^*) \in M \times U$ of Problem 4.1.

**Proof. Step 1:** Construct a minimizing sequence: There exists an infimum $F^*$ such that

$$0 \leq F^* = \inf_{(m, u) \in M \times U} F(m, u) \quad \text{subject to} \quad (3.2),$$

since by Lemma 3.3 the set of solutions for (3.2) is nonempty. So we can choose a minimizing sequence $\{(m_l, u_l)\}_{l \geq 0}$ such that $F(m_l, u_l) \rightarrow F^*$ for $l \rightarrow \infty$.

**Step 2:** Construct a potential minimum $(m^*, u^*)$: Since $F$ is coercive on $L^2(L^2) \times L^2(H^1)$, the minimizing sequence is bounded,

$$\|m_l\|_{L^2(L^2)} + \|u_l\|_{L^2(H^1)} \leq C.$$ 

In particular, the controls $\{u_l\}$ are bounded independently of $l$, such that we get with Lemma 3.3, Aubin-Lions’ Theorem, [22] Chapter III, Proposition 1.3, and the properties of the spaces
Lemma 4.4. We now check the assumptions which are needed to apply the Lagrange multiplier theorem. The existence of \( m^* \in L^2(\mathcal{H}^2) \cap L^\infty(\mathcal{H}^1) \cap H^1(\mathcal{L}^2) \) and \( u^* \in L^2(\mathcal{H}^1) \), as well as subsequences \( \{(m_i, u_i)\} \) (not relabelled) for \( l \to \infty \), such that

\[
\begin{align*}
  u_i &\to u^* \quad \text{weakly in } L^2(\mathcal{H}^1), \\
  m_i &\to m^* \quad \text{weakly in } L^2(\mathcal{H}^2), \\
  (m_i)_t &\to m^*_t \quad \text{weakly in } L^2(\mathcal{L}^2), \\
  m_i &\to^{*} m^* \quad \text{weakly-star in } L^\infty(\mathcal{H}^1), \\
  m_i &\to m^* \quad \text{strongly in } L^2(\mathcal{H}^1).
\end{align*}
\]

**Step 3:** Identify \((m^*, u^*)\) as a solution of (3.2): It is not difficult to identify the limit of each (in particular nonlinear) term with its counterpart in (3.2), using smooth test functions \( \varphi \in C^\infty(\mathcal{C}^\infty) \).

For the initial condition we infer for \( \varphi \in C^\infty(\mathcal{C}^\infty) \) with \( \varphi(T) = 0 \), that for \( l \to \infty \)

\[
\begin{align*}
  \int_0^T (m_i^*, \varphi) \, ds - \int_0^T ((m_i)_t, \varphi) \, ds &= - \int_0^T (m_i, \varphi_i) \, ds - (m_i(0), \varphi(0)) \\
  &\to - \int_0^T (m^*, \varphi_i) \, ds - (m_0, \varphi(0)),
\end{align*}
\]

hence \( m^*(0) = m_0 \).

**Step 4:** \((m^*, u^*)\) is a minimum: Using that \( F \) is weakly lower semicontinuous leads to

\[ F(m^*, u^*) = F^*. \]

\[ \square \]

4.2. **Optimality system.** In order to deduce the necessary optimality system for a minimum of Problem 4.1 we use the following reformulation, Problem 4.3 and the Lagrange multiplier theorem [16, Chapter 9, Theorem 1]. To this end we define the two functions \( e : M \times U \to L^2(\mathcal{L}^2) \) and \( a : M \times U \to H^1(\Omega) \) by

\[
\begin{align*}
  e(m, u) &:= m - \alpha \Delta m - \alpha |\nabla m|^2 m + \alpha m \times (m \times u) - m \times (\Delta m + u), \\
  a(m, u) &:= m(0) - m_0.
\end{align*}
\]

Then we may state Problem 4.1 in the following form.

**Problem 4.3.** Let \( \bar{m} : [0, T] \times \Omega \to S^2 \) be smooth and given, as well as \( m_0 \in H^2(\Omega) \) with \( |m_0|^2 = 1 \) in \( \Omega \) and \( \lambda_0, \lambda_1 > 0 \). Minimize \( F \) subject to \( H(m, u) = 0 \), where

\[
H : M \times U \to L^2(\mathcal{L}^2) \times H^1(\Omega), \quad H(m, u) := \begin{pmatrix} e(m, u) \\ a(m, u) \end{pmatrix}.
\]

We now check the assumptions which are needed to apply the Lagrange multiplier theorem.

**Lemma 4.4.** The function \( H : M \times U \to L^2(\mathcal{L}^2) \times H^1(\Omega) \) is continuously Fréchet differentiable, with derivative

\[
\langle H'(m, u), (\delta m, \delta u) \rangle = \begin{pmatrix} \langle e'(m, u), (\delta m, \delta u) \rangle \\ \langle a'(m, u), (\delta m, \delta u) \rangle \end{pmatrix}.
\]
where
\[
\langle e'(m, u), (\delta m, \delta u) \rangle = \delta m_t - \alpha \Delta m - \alpha |\nabla m|^2 \delta m - 2\alpha \langle \nabla m, \nabla \delta m \rangle m \\
+ \alpha \delta m \times (m \times u) + \alpha m \times (\delta m \times u) + \alpha m \times (m \times \delta u) \\
- \delta m \times \Delta m - m \times \Delta \delta m - \delta m \times u - m \times \delta u,
\]
\[
\langle a'(m, u), (\delta m, \delta u) \rangle = \delta m(0).
\]

Proof. See [21] Lemmas 2.5, 2.6.

Next, we show that minima of Problem 4.3 are regular points.

**Lemma 4.5.** Let \((m^*, u^*)\) be a minimum of Problem 4.3. Then it is a regular point of the function \(H : M \times U \to L^2(L^2) \times H^1(\Omega)\).

Proof. It is enough to show that for a fixed \(u \in U\), e.g. \(u = 0\),
\[
(v, 0) \mapsto \langle H'(m^*, u^*), (v, 0) \rangle : M \times U \to L^2(L^2) \times H^1(\Omega)
\]
is surjective. It is easy but technical to prove that the map is surjective, where \((m^*, u^*)\) fulfills (3.1): make a semi-discretization in time with a semi-implicit Euler method (analogously to the semi-discrete version, cf. Lemma 6.6) and show its surjectivity by means of the Lax-Milgram lemma. Then pass to the limit afterwards, see [21] Lemma 2.7.

Because of Lemmas 4.4 and 4.5, all assumptions in the Lagrange multiplier theorem in [16] Chapter 9, Theorem 1 are fulfilled; hence, there exists \((z, \zeta) \in L^2(L^2) \times H^1(\Omega)^*\), such that the Lagrange functional \(L : M \times U \to \mathbb{R}\) with
\[
L(m, u) := F(m, u) + \langle (z, \zeta), H(m, u) \rangle = F(m, u) + \langle z, e(m, u) \rangle + \langle \zeta, a(m, u) \rangle
\]
is stationary at a minimum \((m^*, u^*)\), so
\[
DL(m^*, u^*) = DF(m^*, u^*) + \langle z, De(m^*, u^*) \rangle + \langle \zeta, Da(m^*, u^*) \rangle = 0.
\]
The directional derivatives are already computed in Lemma 4.4 and we get in combination with the state equation the necessary optimality system for a minimum \((m^*, u^*)\) of Problem 4.1
\[
0 = m_t^* - \alpha \Delta m^* - \alpha |\nabla m^*|^2 m^* + \alpha m^* \times (m^* \times u^*) - m^* \times (\Delta m^* + u^*), \tag{4.1a}
\]
\[
0 = \lambda_0 u^* - \lambda_1 \Delta u^* + \alpha (z \times m^*) \times m^* - z \times m^*, \tag{4.1b}
\]
\[
0 = - (z_t, \delta m) - \alpha (z, \Delta \delta m) - (\delta m - m^*, \delta m) - \alpha (z, |\nabla m^*|^2 \delta m) \\
- 2\alpha (z, (\nabla m^*, \nabla \delta m) m^*) + \alpha (z, \delta m \times (m^* \times u^*)) + \alpha (z, m^* \times (\delta m \times u^*)) \\
- (z, m^* \times \Delta \delta m) - (z, \delta m \times \Delta m^*) - (z, \delta m \times u^*) \tag{4.1c}
\]
for almost all \(t > 0\) and for all \(\delta m \in M\), together with \(m^*(0) = m_0\) and \(z(T) = 0\).

Here, (4.1a) together with the initial condition \(m^*(0) = m_0\) is the state equation (3.2), and the optimality condition (4.1b) follows from the directional derivative of \(L\) in \(u\) in a strong formulation. The adjoint equation (4.1c) with terminal condition \(z(T) = 0\) is obtained analogously to [10] Section 2.6 by a standard argument from the directional derivative of \(L\) in \(m\).
Lemma 4.7. Let \( \{m^*, u^*\} \) be a solution of Problem 4.1. Then the solution \( z \) of the adjoint equation (4.1c) satisfies
\[
z \in L^2(H^2) \cap L^\infty(H^1) \cap H^1(L^2) \hookrightarrow C(H^1) \hookrightarrow L^\infty(L^\infty)
\]
and the optimal control \( u^* \) satisfies
\[
u^* \in L^\infty(H^2) \cap H^1(H^1) \hookrightarrow C(H^1).
\]

Proof. The tuple \( \{m^*, u^*\} \) is a solution of Problem 4.1 and hence fulfills the state equation (3.1). By Lemmas 3.3 and 3.4 we may infer the regularity properties
\[
m^* \in L^2(H^3) \cap C(H^2) \cap H^1(H^1), \quad u^* \in L^2(H^1).
\]
Test the adjoint equation (4.1c) formally with \( z, -\Delta z \) and use integration by parts and \( H^1(\Omega) \hookrightarrow L^\infty(\Omega) \) to bound the terms separately. The terminal condition \( z(T) = 0 \) and the Gronwall lemma then show the assertion.

Multiply the optimality condition (4.1b) with \( u^*, -\Delta u^* \), and the formal time derivative of the optimality condition,
\[
\lambda_0 u^*_t - \lambda_1 \Delta u^*_t = -\alpha(z_t \times m^*) \times m^* - \alpha(z \times m^*_t) \times m^*
\]
with \( u^*_t \) and use the regularity of the adjoint \( z \) to show the statement. \( \square \)

5. Semi-discretization in time of the state equation

Let \( k > 0 \) be the equi-distant size of a mesh \( t_j = \{t_j\}_{j=0}^J \) which covers \([0,T]\). In this section we consider the semi-discrete state equation. First we will show the existence of a solution and then deduce regularity properties for the semi-discrete state. Both properties are linked which is why we proceed for every iteration index \( j \geq 0 \); see the use of Corollary 5.6 in the proof of Lemma 5.2.

Given a fixed control \( \{u^j\}_{j=1}^J \subseteq H^1(\Omega) \), the semi-discrete magnetization \( \{m^j\}_{j=0}^J \) fulfills the following semi-implicit scheme
\[
d_t m^{j+1} - \alpha \Delta m^{j+1} = \alpha |\nabla m^j|^2 m^{j+1} - \alpha m^{j+1} \times (m^j \times u^{j+1}) \]
\[
+ m^{j+1} \times \Delta m^{j+1} + m^{j+1} \times u^{j+1}
\]
(5.1)
with \( m^0 = m_0 \).

Definition 5.1 (Strong solution). Let the control \( \{u^j\}_{j=1}^J \subseteq H^1(\Omega) \). We call \( \{m^j\}_{j=0}^J \) a strong solution of (5.1) if
(1) \( m^j \in H^2(\Omega) \) for all \( j = 0, \ldots, J \),
(2) for all \( j = 0, \ldots, J - 1 \) and for all \( \varphi \in C^\infty(\Omega) \) holds
\[
\left( d_m m^{j+1}, \varphi \right) + \alpha \left( \nabla m^{j+1}, \nabla \varphi \right) = \alpha \left( |\nabla m_j|^2 m^{j+1}, \varphi \right) - \alpha \left( m^{j+1} \times (m^j \times u^{j+1}), \varphi \right) \\
- \left( m^{j+1} \times \nabla m^{j+1}, \nabla \varphi \right) + \left( m^{j+1} \times u^{j+1}, \varphi \right)
\]
(3) \( m^0 = m_0 \in H^2(\Omega) \).

Notice that we do not claim in the Definition 5.1 that the iterates of the semi-discretization take values on the sphere or fulfill an energy equality, as it was done in Definition 3.1 for the strong solution of the LLG equation (3.1). According to the argumentation in [18, Example 4.2] or numerical experiments, iterates are in general not \( S^2 \)-valued, which prevents to show an energy equality as for the continuous counterpart.

First we have to show the existence of strong solutions of this scheme, which requires that time steps \( k \leq k_0 \left( T, \|m_0\|_{H^2(\Omega)} \right) \) have to be sufficiently small.

**Lemma 5.2.** Let \( T > 0 \) and \( m_0 \in H^2(\Omega) \) with \( |m_0|^2 = 1 \) in \( \Omega \). For every \( k \geq 0 \), let \( J = \frac{T}{k} \), and \( \{u^j\}_{j=1}^J \) be a semi-discrete control, such that \( k \sum_{j=1}^J \|u^j\|_{H^1(\Omega)}^2 \leq K \), where \( K \) is independent of \( k \). Then there is a number \( k_0 = k_0 \left( \Omega, T, \alpha, K, \|m_0\|_{H^2(\Omega)} \right) > 0 \), such that a unique strong solution \( \{m^j\}_{j=0}^J \subseteq H^2(\Omega) \) of (5.1) exists for all time steps \( k \leq k_0 \).

**Proof.** We proceed by induction to show the existence of a new iterate \( m^{j+1} \in H^2(\Omega) \) which solves (5.1): the uniform bounds from Corollary 5.6 guarantee that this result holds for \( k_0 \) sufficiently small.

**Step 1:** Existence of a smooth approximate via a Galerkin ansatz in space: Let \( \{\varphi_l\}_{l \in \mathbb{N}} \) be an orthonormal basis of \( L^2(\Omega) \) with \( \varphi_l \in C^\infty(\Omega) \) and \( V_N = \text{span}\{\varphi_l\}_{l=1}^N \). Then \( m_N^{j+1} := \sum_{l=1}^N b_l \varphi_l \). Now we consider the discretization of (5.1) in space:

Find for \( l = 1, \ldots, N \) values \( b_l \in \mathbb{R} \), such that for all \( \varphi \in V_N \)
\[
\left( \frac{1}{k} m_N^{j+1} \right) l \varphi_l \right) + \alpha \left( \nabla m_N^{j+1}, \nabla \varphi \right) = \alpha \left( |\nabla m_N|^2 m_N^{j+1}, \varphi \right) - \alpha \left( m_N^{j+1} \times (m^j \times u^{j+1}), \varphi \right) \\
- \left( m_N^{j+1} \times \nabla m_N^{j+1}, \nabla \varphi \right) + \left( m_N^{j+1} \times u^{j+1}, \varphi \right),
\]
where \( m_N^0 := m^0 \).

Solvability for this scheme now follows by a consequence of the Brouwer fixed-point theorem, see [20], Lemma 2.26: we show that there is a finite \( R > 0 \), such that \( \sum_{l=1}^N g_l(b) b_l \geq 0 \) for all \( b \) with \( |b| = R \), where
\[
g_l(b) := \left( \frac{1}{k} \sum_{l=1}^N b_l \varphi_l - \frac{1}{k} m^j, \varphi_l \right) + \alpha \left( \sum_{l=1}^N b_l \nabla \varphi_l, \nabla \varphi_l \right) - \alpha \left( |\nabla m^j|^2 \sum_{l=1}^N b_l \varphi_l, \varphi_l \right) \\
+ \alpha \left( \sum_{l=1}^N b_l \varphi_l \times (m^j \times u^{j+1}), \varphi_l \right)
\]
for \( i = 1, \ldots, N \). By the \( L^2(\Omega) \)-orthonormality of the \( \{ \varphi_i \}_{i \in N} \), the properties of the vector product and an interpolation of \( L^\infty(\Omega) \) between \( L^2(\Omega) \) and \( H^1(\Omega) \), we find
\[
\sum_{i=1}^N g_i(b_i) b_i \geq \frac{1}{2k} \left( \sum_{i=1}^N b_i \varphi_i \right)^2 - \frac{1}{2k} \left\| m^j \right\|_{L^2(\Omega)}^2 + \frac{1}{2k} \left( \sum_{i=1}^N b_i \varphi_i \right)^2 - \alpha \left\| \nabla m^j \right\|_{L^2(\Omega)}^2 \sum_{i=1}^N b_i \varphi_i \left\| m^j \right\|_{L^2(\Omega)} \sum_{i=1}^N b_i \varphi_i \right\|_{L^\infty(\Omega)} \right).
\]
By means of Corollary \ref{cor:4} we may choose \( R = 1 \) and \( k_0 = \left( \Omega, T, \alpha, K, \| m_0 \|_{H^2(\Omega)} \right) \) small enough and independently of \( j \) and the time step \( k \) to conclude \( \sum_{i=1}^N g_i(b_i) b_i \geq 0 \).

**Step 2:** Definition of a candidate \( m_{s+1}^j \): Again, we may use Corollary \ref{cor:4} for \( k_0 \) small enough to get for a fixed \( k > 0 \) the bound \( \left\| m_{s+1}^j \right\|_{H^2(\Omega)} \leq C \), where \( C \) does not depend on \( N \). So there exist a \( m_{s+1}^j \in H^2(\Omega) \) and a subsequence \( m_{s+1}^N \), such that for \( N \to \infty \) the subsequence satisfies
\[
m_{s+1}^N \rightharpoonup m_{s+1}^j \quad \text{weakly in } H^2(\Omega),
m_{s+1}^N \to m_{s+1}^j \quad \text{strongly in } H^1(\Omega).
\]

**Step 3:** \( m_{s+1}^j \in H^2(\Omega) \) fulfills the equation \( \mathcal{A}^j + \mathcal{B}(\cdot, m^j) = 0 \) in the \((j+1)\)th step: We may easily identify the weak limit in each term with the help of the bounds and convergence properties of the iterates \( m_{s+1}^j \) given in Step 2, using smooth test functions \( \varphi \in C^\infty(\Omega) \).

**Step 4:** Uniqueness of the strong solution: Consider the error equation of two possible solutions and get for \( k_0 \) sufficiently small the uniqueness with the help of the discrete Gronwall lemma and the bounds of the iterates which are given in Corollary \ref{cor:4} and Lemma \ref{lem:5}.

\[ \square \]

**Remark 5.3** (Semi-discrete controls \( \{ u^j \}_{j=1}^J \subseteq H^1(\Omega) \)). By Lemma \ref{lem:2}, a unique strong solution of the continuous state equation \( \mathcal{A}^j + \mathcal{B}(\cdot, m^j) = 0 \) exists already for \( u \in L^2(\mathcal{L}^2) \). For the semi-discretization it is not clear whether a solution of \( \mathcal{A}^j + \mathcal{B}(\cdot, m^j) = 0 \) exists for \( \{ u^j \}_{j=1}^J \subseteq L^2(\mathcal{L}^2) \).

Next, we prove stability properties for the iterates of \( \mathcal{A}^j + \mathcal{B}(\cdot, m^j) = 0 \). The problem to accomplish this goal is that the iterates \( \{ m^j \}_{j=0}^J \) do not preserve the length and so it is not clear how to get a discrete energy estimate. For applied forces \( u \in L^2(\mathcal{L}^1) \) we may benefit from the better regularity properties of the state \( m \) which solves \( \mathcal{A}^j + \mathcal{B}(\cdot, m^j) = 0 \), cf. Lemma \ref{lem:3}, to run an inductive perturbation argument and use a variational argument as in \ref{ineq:2} to show bounds for the iterates \( \{ m^j \}_{j=0}^J \). The error estimates in Theorem \ref{thm:5} for \( \{ e^j \}_{j=0}^J \), where \( e^j := m(t_j) - m^j \), will be used in Corollary \ref{cor:5} to deduce the stability properties for the iterates of \( \mathcal{A}^j + \mathcal{B}(\cdot, m^j) = 0 \).
Theorem 5.4. Let $T > 0$ and $m_0 \in H^2(\Omega)$ with $|m_0|^2 = 1$ in $\Omega$. For every $k \geq 0$, let $J = \frac{T}{k}$, and $\{u^j\}_{j=1}^J$ be a semi-discrete control, such that $k \sum_{j=1}^J \|u^j\|^2_{H^1(\Omega)} \leq K$, where $K$ is independent of $k$. Let $m : [0, T] \times \Omega \to \mathbb{S}^2$ be the strong solution of (3.1) with control $U^+ : [0, T] \times \Omega \to \mathbb{R}^3$ and initial value $m(0) = m_0$, and let $\{m^j\}_{j=0}^J$ be the semi-discrete solution of (5.1) with control $\{u^j\}_{j=1}^J$ and initial condition $m^0$. There exist two constants $k_0 = k_0(\Omega, T, \alpha, K, \|m_0\|_{H^2(\Omega)}) > 0$ and $C = C(\Omega, T, \alpha, K, \|m_0\|_{H^2(\Omega)})$, such that for $k \leq k_0$

\[
\max_{0 \leq j \leq J-1} \left\| e^{j+1} \right\|^2_{L^2(\Omega)} + \frac{\alpha}{4} k \sum_{j=0}^l \left\| \nabla e^{j+1} \right\|^2_{L^2(\Omega)} + \frac{\alpha}{8} k^2 \sum_{j=0}^l \left\| \Delta m^{j+1} \right\|^2_{L^2(\Omega)} \leq kC_1 e^{C_2 t},
\]

Proof. The proof is long and technical. In order to highlight main steps, we skip some detailed estimations of terms and therefore refer to [21, Theorem 3.2].

Write $C(m) := C(K, \|m_0\|_{H^2(\Omega)}, T)$.

Step 1: Definition and set-up of the perturbed equation: We use the notation which is introduced in Section 2 to restate (5.1) as follows,

\[
\mathcal{M}_t = \alpha \Delta \mathcal{M}^+ + \alpha |\nabla \mathcal{M}^-|^2 \mathcal{M}^+ - \alpha \mathcal{M}^+ \times (\mathcal{M}^- \times U^+) + \mathcal{M}^+ \times \Delta \mathcal{M}^+ + \mathcal{M}^+ \times U^+ =: \sum_{i=1}^5 B_i(\mathcal{M}^-, \mathcal{M}^+, U^+) =: B(\mathcal{M}^-, \mathcal{M}^+, U^+).
\]

We use the same terminology for the continuous LLG equation (3.2),

\[
m_t = B(m, m, U^+).
\]

The fundamental theorem of calculus and the definitions (5.2) and (5.3) lead to

\[
\|m(t) - \mathcal{M}(t)\|^2_{L^2(\Omega)} = \int_0^t \frac{d}{ds} \|m(s) - \mathcal{M}(s)\|^2_{L^2(\Omega)} \, ds
\]

\[
= 2 \int_0^t \left( m - \mathcal{M}^+, B(m, m, u) - B(\mathcal{M}^+, \mathcal{M}^-, U^+) \right) \, ds
\]

\[
+ 2 \int_0^t \left( \mathcal{M}^+ - \mathcal{M}, B(m, m, u) - B(\mathcal{M}^+, \mathcal{M}^-, U^+) \right) \, ds
\]

\[=: 2A_1 + 2A_2.
\]

We use this error equation in the following Steps 3 to 5 to allow for an inductive argument in Step 6, which verifies the following: There exist constants $C_i = C_i(\Omega, T, \alpha, m, U^+)$, $i = 1, 2$, such that we have for $0 \leq l \leq J - 1$

\[
\max_{0 \leq j \leq l} \left\| e^{j+1} \right\|^2_{L^2(\Omega)} + \frac{\alpha}{4} k \sum_{j=0}^l \left\| \nabla e^{j+1} \right\|^2_{L^2(\Omega)} + \frac{\alpha}{8} k^2 \sum_{j=0}^l \left\| \Delta m^{j+1} \right\|^2_{L^2(\Omega)} \leq kC_1 e^{C_2 t},
\]
in particular
\[
\max_{0 \leq j \leq l} \left\| \nabla e^{j+1} \right\|^2_{L^2(\Omega)} \leq C I e^{C_2 t_1}.
\]

We will see that the terms in \( A_1 \) are more difficult to handle than the terms in \( A_2 \), because every term there has a small factor of order \( \mathcal{O}(k) \) after a reformulation; cf. Step 4.

**Step 2:** Tools: According to Lemmas 3.3 and 3.4 we have \( m \in C(H^2) \cap H^1(H^1) \), and therefore \( m^+, \nabla m^+, \Delta m^+ \), and \( m^-, \nabla m^-, \Delta m^- \) are well-defined and satisfy
\[
\|m\|_{L^\infty(H^2)} + \|m^+\|_{L^\infty(H^2)} + \|m^-\|_{L^\infty(H^2)} \leq C(m).
\]
Moreover,
\[
\|m - m^+\|_{L^\infty(H^1)}^2 + \|m - m^-\|_{L^\infty(H^1)}^2 \leq k C(m).
\]

We frequently use the following Gagliardo-Nirenberg inequalities in one dimension,
\[
\|m\|_{L^4(\Omega)} \leq C \|m\|_{H^1(\Omega)}^{\frac{1}{2}} \|m\|_{L^2(\Omega)}^{\frac{1}{2}}, \quad \|m\|_{L^\infty(\Omega)} \leq C \|m\|_{H^1(\Omega)}^{\frac{1}{2}} \|m\|_{L^2(\Omega)}^{\frac{1}{2}}.
\]

**Step 3:** Bounds for \( A_1 \): By the definition of \( B \) in (5.2) and (5.3), respectively, we have
\[
A_1 = \sum_{i=1}^5 \int_0^t \left( m - \mathcal{M}^+, B_i(m, m, \mathcal{U}^+) - B_i(\mathcal{M}^-, \mathcal{M}^+, \mathcal{U}^+) \right) ds =: \sum_{i=1}^5 A_{1i}.
\]
We estimate the terms separately with the help of the tools in Step 2: Let \( \sigma > 0 \).

For \( A_{11} \), we use integration by parts and the reverse triangle inequality to conclude as follows
\[
A_{11} = -\alpha \int_0^t \left\| \nabla m \pm \nabla m^+ - \nabla \mathcal{M}^+ \right\|^2_{L^2(\Omega)} ds
\leq -\frac{\alpha}{2} \int_0^t \left\| \nabla m^+ - \nabla \mathcal{M}^+ \right\|^2_{L^2(\Omega)} ds + C \int_0^t \left\| \nabla m - \nabla m^+ \right\|^2_{L^2(\Omega)} ds
\leq -\frac{\alpha}{2} \int_0^t \left\| \nabla \mathcal{E}^+ \right\|^2_{L^2(\Omega)} ds + k C(m),
\]
where \( \mathcal{E}^+ \) and (in the following) \( \mathcal{E} \) and \( \mathcal{E}^- \), respectively, describe the piecewise affine and constant interpolants of the error \( \{e^j\}_{j=0} \), respectively. So the term \( \alpha \int_0^t \left\| \nabla \mathcal{E}^+(s) \right\|^2_{L^2(\Omega)} ds \) can be used later to absorb terms. For the next term \( A_{12} \) we may use the identity
\[
\langle \nabla e^j, 2 \nabla m(t_j) - \nabla e^j \rangle = |\nabla m(t_j)|^2 - |\nabla m^j|^2
\]
to rewrite it as follows,
\[
A_{12} = \alpha \int_0^t \left( m - \mathcal{M}^+, |\nabla m|^2 m - |\nabla m^-|^2 m^+ \right) ds + \alpha \int_0^t \left( m - \mathcal{M}^+, |\nabla m^-|^2 \mathcal{E}^+ \right) ds
+ \alpha \int_0^t \left( m - \mathcal{M}^+, \langle \nabla \mathcal{E}^-, 2 \nabla m^- - \nabla \mathcal{E}^- \rangle \mathcal{M}^+ \right) ds
=: a + b + c.
\]
The strategy here is again to benefit from the interpolation error of $m$. We estimate each term separately and may use the tools in Step 2 to bound

$$a + b \leq kC(m) + C(m) \int_0^t \|E^+\|^2_{L^2(\Omega)} \, ds.$$  

The term $c$ is split into several parts,

$$\frac{1}{\alpha} c = \int_0^t \left( m + m^+ - \mathcal{M}^+, (\nabla E^-, 2\nabla m^-)m^+ \right) \, ds - \int_0^t \left( m + m^+ - \mathcal{M}^+, |\nabla E^-|^2 m^+ \right) \, ds$$

$$- \int_0^t \left( m + m^+ - \mathcal{M}^+, (\nabla E^-, 2\nabla m^-)E^+ \right) \, ds + \int_0^t \left( m + m^+ - \mathcal{M}^+, |\nabla E^-|^2 E^+ \right) \, ds$$

$$=: c_1 + \ldots + c_4.$$  

We estimate the single terms, using the Gagliardo-Nirenberg inequality to preserve parts of the $L^2(\Omega)$ norm in terms $c_2$ to $c_4$. We get

$$c_1 \leq kC(m) + C(\sigma, m) \int_0^t \|E^+\|^2_{L^2(\Omega)} \, ds + \sigma \int_0^t \|\nabla E^-\|^2_{L^2(\Omega)} \, ds,$$

$$c_2 \leq kC(\sigma, m) \int_0^t \|\nabla E^-\|^2_{L^2(\Omega)} \, ds + \sigma \int_0^t \|\nabla E^-\|^2_{L^2(\Omega)} \, ds$$

$$+ C(\sigma, m) \int_0^t \left[ 1 + \|\nabla E^-\|^4_{L^2(\Omega)} \right] \|E^+(s)\|^2_{L^2(\Omega)} \, ds + \sigma \int_0^t \|\nabla E^+(s)\|^2_{L^2(\Omega)} \, ds,$$

$$c_3 \leq kC(\sigma, m) + C(\sigma) \int_0^t \left[ 1 + \|\nabla E^-\|^8_{L^2(\Omega)} \right] \|E^+\|^2_{L^2(\Omega)} \, ds + \sigma \int_0^t \|\nabla E^+\|^2_{L^2(\Omega)} \, ds,$$

$$c_4 \leq kC(\sigma, m) + C(\sigma) \int_0^t \left[ 1 + \|\nabla E^-\|^8_{L^2(\Omega)} \right] \|E^+\|^2_{L^2(\Omega)} \, ds + \sigma \int_0^t \|\nabla E^+\|^2_{L^2(\Omega)} \, ds.$$  

Note that we can absorb $\sigma \int_0^t \|\nabla E^-\|^2_{L^2(\Omega)} \, ds$, because $\nabla E^-(s) = 0$ for $s \in (0, k]$. We use for the next three terms the properties of the vector product, and for $A_{14}$ integration by parts to get

$$A_{13} = \alpha \int_0^t \left( E^+, m \times (E^- \times U^+) \right) \, ds + \alpha \int_0^t \left( E^+, m \times ((m - m^-) \times U^+) \right) \, ds$$

$$+ \alpha \int_0^t \left( m - m^+, m \times (E^- \times U^+) \right) \, ds + \alpha \int_0^t \left( m - m^+, m \times ((m - m^-) \times U^+) \right) \, ds$$

$$\leq C(m) \left( \int_0^t \|E^+\|^2_{L^2(\Omega)} \, ds + \int_0^t \|U^+\|^2_{L^\infty(\Omega)} \|E^-\|^2_{L^2(\Omega)} \, ds + k \int_0^t \|U^+\|^2_{L^2(\Omega)} \, ds \right),$$

$$A_{14} = - \int_0^t \left( m + m^+ - \mathcal{M}^+, \nabla m \times (\nabla m \pm \nabla m^+ - \nabla \mathcal{M}^+) \right) \, ds$$

$$\leq kC(m) + C(\sigma, m) \int_0^t \|E^+\|^2_{L^2(\Omega)} \, ds + \sigma \int_0^t \|\nabla E^+\|^2_{L^2(\Omega)} \, ds.$$  

Finally, we compute $A_{15} = 0$.

**Step 4:** Bounds for $A_2$: The following identity is valid for $s \in (t_j, t_{j+1})$,

$$\mathcal{M}^+(s) - \mathcal{M}(s) = (t_{j+1} - s)B(\mathcal{M}^-, \mathcal{M}^+, U^+),$$

because

$$\mathcal{M}(s) = \frac{s - t_{j+1}}{k}(\mathcal{M}^+ - \mathcal{M}^-) + \mathcal{M}^+ = (s - t_{j+1})\mathcal{M}_t + \mathcal{M}^+.$$
As a consequence, the term $A_2$ may be rephrased as

$$A_2 = \int_0^t (s^+ - s) \left( B(\mathcal{M}^-, \mathcal{M}^+, \mathcal{U}^+), B(m, m, \mathcal{U}^+) - B(\mathcal{M}^-, \mathcal{M}^+, \mathcal{U}^+) \right) \, ds,$$

where $s^+$ is the piecewise constant time interpolation of $\{t_j\}_{j=0}^J$. Now we have 25 terms to bound, which will be numbered by $A_{2ij}$. This means, that the first argument is the term $B_i(\mathcal{M}^-, \mathcal{M}^+, \mathcal{U}^+)$ and the second argument is the term $B_j(m, m, \mathcal{U}^+) - B_j(\mathcal{M}^-, \mathcal{M}^+, \mathcal{U}^+)$. In the following let $t = t_l$ for $1 \leq l \leq J$.

There are two useful properties: The first is the bound $s^+ - s \leq k$ and so every term $A_{2ij}$ is of order $O(k)$; second, there are terms from $A_{2ii}$ having the right sign, which we explain now.

Consider first of all the terms of the form $A_{2ii}$. Employ that $B_i(\mathcal{M}^-, \mathcal{M}^+, \mathcal{U}^+)$ is piecewise constant, and $\int_{t_{j-1}}^{t_j} (s^+ - s) \, ds = \frac{1}{2} k^2$ to get

$$A_{2ii} = \int_0^t (s^+ - s) \left( B_i(\mathcal{M}^-, \mathcal{M}^+, \mathcal{U}^+), B_i(m, m, \mathcal{U}^+) \right) \, ds$$

$$- \sum_{j=1}^l \left\| B_i(m^{j-1}, m^j, \mathcal{U}^+) \right\|^2_{L^2(\Omega)} \int_{t_{j-1}}^{t_j} (s^+ - s) \, ds$$

$$\leq k \int_0^t \left\| B_i(m, m, \mathcal{U}^+) \right\|^2_{L^2(\Omega)} \, ds - \frac{1}{4} k \int_0^t \left\| B_i(\mathcal{M}^-, \mathcal{M}^+, \mathcal{U}^+) \right\|^2_{L^2(\Omega)} \, ds$$

$$\leq kC(m) - \frac{1}{4} k \int_0^t \left\| B_i(\mathcal{M}^-, \mathcal{M}^+, \mathcal{U}^+) \right\|^2_{L^2(\Omega)} \, ds.$$

Now we have with $A_{211}$ to $A_{255}$ the following positive terms on the left hand side:

$$\frac{1}{4} k \int_0^t \left\| \Delta \mathcal{M}^+ \right\|^2_{L^2(\Omega)} \, ds, \quad \frac{1}{4} k \int_0^t \left\| \nabla \mathcal{M}^- \right\|^2_{L^2(\Omega)} \, ds, \quad \frac{1}{4} k \int_0^t \left\| \nabla \mathcal{M}^+ \right\|^2_{L^2(\Omega)} \, ds,$$

$$\frac{1}{4} k \int_0^t \left\| \mathcal{M}^+ \times \mathcal{U}^+ \right\|^2_{L^2(\Omega)} \, ds, \quad \frac{1}{4} k \int_0^t \left\| \mathcal{M}^+ \times (\mathcal{M}^- \times \mathcal{U}^+) \right\|^2_{L^2(\Omega)} \, ds.$$

We have to estimate the remaining terms. The term $A_{212}$ is split into parts like $A_{112}$ in Step 3. By similar calculations, the tools from Step 2, we arrive at the following estimate,

$$A_{212} \leq \sigma k \int_0^t \left\| \Delta \mathcal{M}^+ \right\|^2_{L^2(\Omega)} \, ds + kC(\sigma, m)$$

$$+ C(\sigma, m) k \int_0^t \left[ 1 + \left\| \nabla \mathcal{E}^- \right\|^2_{L^2(\Omega)} + \left\| \nabla \mathcal{E}^- \right\|^2_{L^2(\Omega)} + \left\| \Delta \mathcal{M}^- \right\|^2_{L^2(\Omega)} \right] \left\| \mathcal{E}^+ \right\|^2_{L^2(\Omega)} \, ds$$

$$+ C(\sigma, m) k \int_0^t \left\| \nabla \mathcal{E}^- \right\|^2_{L^2(\Omega)} \left\| \Delta \mathcal{M}^- \right\|^2_{L^2(\Omega)} \left\| \mathcal{E}^+ \right\|^2_{L^2(\Omega)} \, ds$$

$$+ C(\sigma, m) k \int_0^t \left[ \left\| \nabla \mathcal{E}^- \right\|^2_{L^2(\Omega)} + \left\| \nabla \mathcal{E}^- \right\|^4_{L^2(\Omega)} \right] \int_0^t \left\| \nabla \mathcal{E}^- \right\|^2_{L^2(\Omega)} \, ds.$$

The remaining terms are not that sophisticated as the above one. For example, we consider $A_{213}$,

$$A_{213} \leq \sigma k \int_0^t \left\| \Delta \mathcal{M}^+ \right\|^2_{L^2(\Omega)} \, ds + C(\sigma, m) k \int_0^t \left\| \mathcal{U}^+ \right\|^2_{L^2(\Omega)} \, ds$$
where we can bound the last term by
\[
k \int_0^t \| \mathcal{M}^\alpha \times (\mathcal{M}^- \times \mathcal{U}^\alpha) \|^2_{L^2(\Omega)} \, ds
\]

Some terms also benefit from the properties of the vector product. We consider the term \(A_{214}\) as an example for this,
\[
A_{214} = \alpha \int_0^t (s^+ - s) \left( \Delta \mathcal{M}^+, \mathbf{m} \times \Delta \mathbf{m} \right) \, ds \leq \sigma k \int_0^t \| \Delta \mathcal{M}^+ \|^2_{L^2(\Omega)} \, ds + kC(m).
\]
All other terms can be bounded similarly. We only mention to following technical calculation,

\[
A_{221} = \alpha^2 \int_0^t (s^+ - s) \left( |\nabla \mathcal{M}^-|^2 \mathcal{M}^+, \Delta \mathbf{m} \right) \, ds - \alpha^2 \int_0^t (s^+ - s) \left( |\nabla \mathbf{m}|^2 \mathcal{M}^+ \right) \, ds + A_{212}.
\]

**Step 5:** Combination of the above estimates: Choose \(\sigma\) small enough and let
\[
A_{j+1} = C(m) + Ck \left\| \mathbf{u}^{j+1} \right\|^2_{H^1(\Omega)},
\]
\[
B_{j+1} = C(m) \left[ \left\| \nabla \mathbf{e}^j \right\|^2_{L^2(\Omega)} + \left\| \nabla \mathbf{e}^j \right\|^2_{L^2(\Omega)} + \left\| \nabla \mathbf{e}^j \right\|^2_{L^2(\Omega)} \right]
+ kC(m) \left[ \left\| \Delta \mathbf{m}^j \right\|^2_{L^2(\Omega)} + \left\| \nabla \mathbf{e}^j \right\|^2_{L^2(\Omega)} + \left\| \Delta \mathbf{m}^j \right\|^2_{L^2(\Omega)} \right]
+ kC(m) \left[ \left\| \mathbf{u}^{j+1} \right\|^2_{H^1(\Omega)} + \left\| \mathbf{e}^j \right\|^2_{L^2(\Omega)} + \left\| \mathbf{u}^{j+1} \right\|^2_{H^1(\Omega)} \right],
\]
\[
N_{j+1} = \left\| \mathbf{u}^{j+1} \right\|^2_{H^1(\Omega)},
\]
\[
C_j = kC(m) + kC \left\| \mathcal{U}^\alpha \right\|^2_{L^2(H^{-1}(\Omega))} =: k\tilde{C},
\]
\[
D_j = kC(m) \left[ 1 + \left\| \nabla \mathbf{e}^- \right\|^2_{L^2(\Omega)} + \left\| \nabla \mathbf{e}^- \right\|^4_{L^2(\Omega)} \right] \int_0^t \left\| \nabla \mathbf{e}^- \right\|^2_{L^2(\Omega)} \, ds.
\]
According to Steps 3 and 4, the following inequality is valid at time point \(t = t_j\):
\[
\left\| \mathbf{e}^j \right\|^2_{L^2(\Omega)} + \frac{\alpha}{4} k \sum_{i=0}^{j-1} \left\| \nabla \mathbf{e}^{i+1} \right\|^2_{L^2(\Omega)} + \frac{\alpha}{8} k^2 \sum_{i=0}^{j-1} \left\| \Delta \mathbf{m}^{i+1} \right\|^2_{L^2(\Omega)}
\leq k \sum_{i=0}^{j-1} \left( A^{i+1} + B^{i+1} \right) \left\| \mathbf{e}^{i+1} \right\|^2_{L^2(\Omega)} + k \sum_{i=0}^{j-2} N^{i+1} \left\| \mathbf{e}^{i+1} \right\|^2_{L^2(\Omega)} + C^j + D^j.
\]

**Step 6:** Inductive argument: We now want to prove (5.4) by an inductive argument.

**Basis:** Let \(l = 0\): Let \(t_j = t_1\) in (5.5) and get with \(\mathbf{e}^0 = \mathbf{e}^- = \mathbf{0}\) in \((0, t_1)\):
\[
\left\| \mathbf{e}^1 \right\|^2_{L^2(\Omega)} + \frac{\alpha}{4} k \left\| \nabla \mathbf{e}^1 \right\|^2_{L^2(\Omega)} + \frac{\alpha}{8} k^2 \left\| \Delta \mathbf{m}^1 \right\|^2_{L^2(\Omega)} \leq kA^1 \left\| \mathbf{e}^1 \right\|^2_{L^2(\Omega)} + k\tilde{C}.
\]
Because \( kA^1 \leq kC(m) + Ck^2 \sum_{j} \| u_j \|_{H^1(\Omega)}^2 = k\tilde{A} \) we can choose \( k_0 (\Omega, T, \alpha, m) \) small enough to absorb this term and so the statement holds for \( l = 0 \) with 
\[
C_1 := 3\tilde{c}e^{2+2\kappa} (\tilde{A} + 1 + K), \quad C_2 := 2\tilde{A}.
\]

**Inductive step:** \( l \to l+1 \): Let \( t_j = t_{l+1} \) in (5.5) and use the induction hypothesis to absorb terms. We get 
\[
\| e^{l+1} \|_{L^2(\Omega)}^2 + \frac{\alpha}{4} k \sum_{j=0}^{l} \| \nabla e^{j+1} \|_{L^2(\Omega)}^2 + \frac{\alpha}{8} k^2 \sum_{j=0}^{l} \| \Delta m^{j+1} \|_{L^2(\Omega)}^2 
\leq k \sum_{j=0}^{l} (A^{j+1} + B^{j+1}) \| e^{j+1} \|_{L^2(\Omega)}^2 + k \sum_{j=0}^{l-1} N^{j+1} \| e^{j+1} \|_{L^2(\Omega)}^2 + k\tilde{C} + D^{l+1}.
\]

By means of the induction hypothesis we can absorb \( D^{l+1} \) for \( k_0 > 0 \) sufficiently small, 
\[
D^{l+1} \leq C(m) k \left[ 1 + C_1 e^{C_2 T} + C_1^2 e^{2C_2 T} \right] \int_{0}^{t_{l+1}} \| \nabla \mathcal{E} \|_{L^2(\Omega)}^2 \, ds,
\]
and similarly we can absorb \( kB^{l+1} + kA^{l+1} \) for \( k_0 \) small enough, so we get 
\[
\frac{1}{2} \| e^{j+1} \|_{L^2(\Omega)}^2 \leq k \sum_{j=0}^{l-1} (A^{j+1} + B^{j+1} + N^{j+1}) \| e^{j+1} \|_{L^2(\Omega)}^2 + k\tilde{C}.
\]

By the discrete Gronwall lemma and the bound from the induction assumption, 
\[
k \sum_{j=0}^{l-1} (A^{j+1} + B^{j+1} + N^{j+1}) 
\leq \tilde{A} t_l + K + kC(m) \left[ C_1 e^{C_2 T} + C_1^2 e^{2C_2 T} + C_1^4 e^{4C_2 T} \right] 
+ \| \Delta m_0 \|_{L^2(\Omega)}^2 \left[ C(m) k^2 + C(m) C_1 e^{C_2 T} k^2 \right] + k \left[ K + K C_1 e^{C_2 T} \right]
\leq \tilde{A} t_l + K + 1,
\]
for \( k_0 \) small enough, we get 
\[
\| e^{l+1} \|_{L^2(\Omega)}^2 \leq 2k\tilde{C} \exp \left( 2k \sum_{j=0}^{l-1} (A^{j+1} + B^{j+1} + N^{j+1}) \right) \leq [2k\tilde{C}e^{2+2K}] e^{2\tilde{A} t_l}
\]
and arrive at 
\[
\frac{1}{2} \| e^{l+1} \|_{L^2(\Omega)}^2 + \frac{\alpha}{4} k \sum_{j=0}^{l} \| \nabla e^{j+1} \|_{L^2(\Omega)}^2 + \frac{\alpha}{8} k^2 \sum_{j=0}^{l} \| \Delta m^{j+1} \|_{L^2(\Omega)}^2 
\leq \max_{0 \leq j \leq l} \| e^{j+1} \|_{L^2(\Omega)}^2 \sum_{j=0}^{l-1} (A^{j+1} + B^{j+1} + N^{j+1}) + k\tilde{C} \leq kC_1 e^{C_2 t_l},
\]
which completes the proof. \( \Box \)

**Remark 5.5.** Theorem 5.4 uses a perturbation argument to verify the same rate \( O(\sqrt{k}) \) as it is used in [19] for iterates of the implicit Euler method to solve the general evolution \( 
\eta_t + A_y \geq 0, \)
where \( A = \partial \varphi + B \) and \( \partial \varphi \) is a subgradient and \( B \) a general monotone operator. We recall
that the goal in [19] is to avoid regularity requirements with respect to temporal derivatives of the solution, and that the error analysis is based on structural assumptions of \( \mathcal{A} \) only.

With the stability result (Lemma 3.4) for the continuous equation (3.2), we get

**Corollary 5.6.** Let \( T > 0 \) and \( \mathbf{m}_0 \in H^2(\Omega) \) with \( |\mathbf{m}_0|^2 = 1 \) in \( \Omega \). For every \( k \geq 0 \), let \( J = \frac{T}{k} \), and \( \{\mathbf{u}^j\}_{j=1}^J \) be a semi-discrete control, such that \( k \sum_{j=1}^{J-1} \|\mathbf{u}^j\|_{H^1(\Omega)}^2 \leq K \), where \( K \) is independent of \( k \). There exists a constant \( k_0 = k_0 \left( \Omega, T, \alpha, K, \|\mathbf{m}_0\|_{H^2(\Omega)} \right) > 0 \), such that for every \( k \leq k_0 \) the solution \( \{\mathbf{m}^j\}_{j=0}^J \) of (5.1) satisfies the following bounds:

\[
\max_{0 \leq j \leq J-1} \left\| \frac{\partial \mathbf{m}^j}{\partial t} \right\|_{L^2(\Omega)}^2 + \sum_{j=0}^{J-1} \left\| \Delta \mathbf{m}^j \right\|_{L^2(\Omega)}^2 \leq C \left( K, \|\mathbf{m}_0\|_{H^2(\Omega)}, T \right).
\]

Recall that iterates of the semi-discretization (5.1) are not supposed to take values in \( \mathbb{S}^2 \). However, we may use the error estimates in Theorem 5.4 to verify that iterates of (5.1) take values which are close to \( \mathbb{S}^2 \).

**Corollary 5.7.** Let \( T > 0 \) and \( \mathbf{m}_0 \in H^2(\Omega) \) with \( |\mathbf{m}_0|^2 = 1 \) in \( \Omega \). For every \( k \geq 0 \), let \( J = \frac{T}{k} \), and \( \{\mathbf{u}^j\}_{j=1}^J \) be a semi-discrete control, such that \( k \sum_{j=1}^{J-1} \|\mathbf{u}^j\|_{H^1(\Omega)}^2 \leq K \), where \( K \) is independent of \( k \). There exists a constant \( k_0 = k_0 \left( \Omega, T, \alpha, K, \|\mathbf{m}_0\|_{H^2(\Omega)} \right) > 0 \), such that for every \( k \leq k_0 \) the solution \( \{\mathbf{m}^j\}_{j=0}^J \) of (5.1) satisfies

\[
\max_{0 \leq j \leq J-1} \left\| 1 - |\mathbf{m}^j| \right\|_{L^2(\Omega)}^2 \leq \sqrt{k}C \left( K, \|\mathbf{m}_0\|_{H^2(\Omega)}, T \right).
\]

**Proof.** Let \( \mathbf{m} \) be the solution of (3.1) for the initial condition \( \mathbf{m}_0 \) and with control \( \mathbf{u}^+ \). Then

\[
\left\| 1 - |\mathbf{m}^j| \right\|_{L^2(\Omega)}^2 = \left\| |\mathbf{m}(t_{j+1})| - |\mathbf{m}(t_j)| \right\|_{L^2(\Omega)}^2
\]

\[
= \left\| (e^{j+1}, e^{j+1} - 2m(t_{j+1})) \right\|_{L^2(\Omega)}^2
\]

\[
\leq 2 \left\| e^{j+1} \right\|_{L^\infty(\Omega)} \left\| e^{j+1} \right\|_{L^2(\Omega)} + C \left\| e^{j+1} \right\|_{L^2(\Omega)} \left\| m(t_{j+1}) \right\|_{L^\infty(\Omega)}
\]

and the statement follows from the continuous embedding \( H^1(\Omega) \hookrightarrow L^\infty(\Omega) \) and Theorem 5.4.

It is now similar to Lemma 3.4 that we can obtain better bounds, which will simplify the analysis in Section 6. The semi-discrete \( L^2(\mathbf{H}^1) \)-bound for the time derivative will guarantee that in Section 7 the piecewise affine and constant interpolants of the state converge to the same limit.

**Lemma 5.8.** Let \( T > 0 \) and \( \mathbf{m}_0 \in H^2(\Omega) \) with \( |\mathbf{m}_0|^2 = 1 \) in \( \Omega \). For every \( k \geq 0 \), let \( J = \frac{T}{k} \), and \( \{\mathbf{u}^j\}_{j=1}^J \) a semi-discrete control, such that \( k \sum_{j=1}^{J-1} \|\mathbf{u}^j\|_{H^1(\Omega)}^2 \leq K \), where \( K \) is independent of \( k \). There exists a constant \( k_0 = k_0 \left( \Omega, T, \alpha, K, \|\mathbf{m}_0\|_{H^2(\Omega)} \right) > 0 \), such that for every \( k \leq k_0 \) the solution \( \{\mathbf{m}^j\}_{j=0}^J \) of (5.1) satisfies the following bounds:
\[
\begin{align*}
\max_{0 \leq j - 1} \left\| m^{j+1} \right\|^2_{H^2(\Omega)} + k \sum_{j=0}^{J-1} \left\| m^{j+1} \right\|^2_{H^0(\Omega)} + k \sum_{j=0}^{J-1} \left\| d_t m^{j+1} \right\|^2_{H^1(\Omega)} & \le C \left( K, \left\| m_0 \right\|_{H^2(\Omega)} ; T \right).
\end{align*}
\]

**Proof. Step 1:** By (5.1), a Gagliardo-Nirenberg inequality and Corollary 5.6 we get for \( k \) small enough
\[
\begin{align*}
k \sum_{j=0}^{J-1} \left\| d_t m^{j+1} \right\|^2_{L^2(\Omega)} & \le C \left( K, \left\| m_0 \right\|_{H^2(\Omega)} ; T \right).
\end{align*}
\]

**Step 2:** Analogously to the proof of Lemma 3.4 we formally differentiate (5.1) in space and get for \( A^j := \nabla m^j \)
\[
\begin{align*}
d_t A^{j+1} - \Delta A^{j+1} = & \ 2\alpha (\nabla A^j, \nabla m^j) m^{j+1} + \alpha |\nabla m^j|^2 A^{j+1} - \alpha A^{j+1} \times (m^j \times u^{j+1}) \\
- & \ \alpha m^{j+1} \times (A^j \times u^{j+1}) - \alpha m^{j+1} \times (m^j \times \nabla u^{j+1}) + \ A^{j+1} \times \Delta \! m^{j+1} \\
+ & \ m^{j+1} \times \Delta A^{j+1} + \ A^{j+1} \times u^{j+1} + m^{j+1} \times \nabla u^{j+1}.
\end{align*}
\]

Multiply (5.6) with \(-\Delta A^{j+1}\), integrate and use the properties of the vector product to find
\[
\begin{align*}
\frac{1}{2} d_t \left\| \nabla A^{j+1} \right\|^2_{L^2(\Omega)} + \alpha \left\| \Delta A^{j+1} \right\|^2_{L^2(\Omega)} & \le 2\alpha \left( |\nabla A^j|^2 A^{j+1} - \Delta A^{j+1} \right) + \alpha \left( |\nabla m^j|^2 A^{j+1} - \Delta A^{j+1} \right) \\
- & \ \alpha (A^{j+1} \times (m^j \times u^{j+1}), -\Delta A^{j+1}) - \alpha (m^{j+1} \times (A^j \times u^{j+1}), -\Delta A^{j+1}) \\
- & \ \alpha (m^{j+1} \times (m^j \times \nabla u^{j+1}), -\Delta A^{j+1}) + (A^{j+1} \times \Delta m^{j+1}, -\Delta A^{j+1}) \\
+ & \ (A^{j+1} \times u^{j+1}, -\Delta A^{j+1}) + (m^{j+1} \times \nabla u^{j+1}, -\Delta A^{j+1}) \\
= & \ : 2\alpha I_1 + \alpha (I_2 + \ldots + I_5) + I_6 + \ldots + I_8.
\end{align*}
\]

Let \( \sigma > 0 \). Estimate the terms and use the Gagliardo-Nirenberg inequality in term \( I_2 \) for \( L^6(\Omega) \) and \( L^\infty(\Omega) \) and in term \( I_6 \) the embedding \( H^1(\Omega) \hookrightarrow L^\infty(\Omega) \),
\[
\begin{align*}
I_1 & \le C(\sigma) \left\| \nabla A^j \right\|^2_{L^2(\Omega)} \left\| \nabla m^j \right\|^2_{L^6(\Omega)} \left\| m^{j+1} \right\|^2_{L^\infty(\Omega)} + \sigma \left\| \Delta A^{j+1} \right\|^2_{L^2(\Omega)} ; \\
I_2 & \le C(\sigma) \left\| \nabla m^j \right\|^4_{L^2(\Omega)} \left\| A^{j+1} \right\|^2_{L^\infty(\Omega)} + \sigma \left\| \Delta A^{j+1} \right\|^2_{L^2(\Omega)} ; \\
I_3 & \le C(\sigma) \left\| \nabla m^j \right\|^6_{L^2(\Omega)} \left\| A^{j+1} \right\|^2_{L^2(\Omega)} \left\| A^{j+1} \right\|^2_{H^1(\Omega)} + C \left\| \nabla m^j \right\|^2_{H^1(\Omega)} + \sigma \left\| \Delta A^{j+1} \right\|^2_{L^2(\Omega)} ; \\
I_4 & \le C(\sigma) \left\| m^{j+1} \right\|^2_{L^\infty(\Omega)} \left\| A^j \right\|^2_{L^2(\Omega)} \left\| u^{j+1} \right\|^2_{L^\infty(\Omega)} + \sigma \left\| \Delta A^{j+1} \right\|^2_{L^2(\Omega)} ; \\
I_5 & \le C(\sigma) \left\| m^{j+1} \right\|^2_{L^\infty(\Omega)} \left\| m^j \right\|^2_{L^\infty(\Omega)} \left\| \nabla u^{j+1} \right\|^2_{L^2(\Omega)} + \sigma \left\| \Delta A^{j+1} \right\|^2_{L^2(\Omega)} ; \\
I_6 & \le C(\sigma) \left\| A^{j+1} \right\|^2_{L^2(\Omega)} \left\| \Delta m^{j+1} \right\|^2_{L^\infty(\Omega)} + \sigma \left\| \Delta A^{j+1} \right\|^2_{L^2(\Omega)} ; \\
I_7 & \le C(\sigma) \left\| A^{j+1} \right\|^2_{L^2(\Omega)} \left\| \Delta \! m^{j+1} \right\|^2_{L^2(\Omega)} + C(\sigma) \left\| A^{j+1} \right\|^4_{L^2(\Omega)} + \sigma \left\| \Delta A^{j+1} \right\|^2_{L^2(\Omega)} ; \\
I_8 & \le C(\sigma) \left\| A^{j+1} \right\|^2_{L^2(\Omega)} \left\| \Delta m^{j+1} \right\|^2_{L^2(\Omega)} + C(\sigma) \left\| A^{j+1} \right\|^4_{L^2(\Omega)} + \sigma \left\| \Delta A^{j+1} \right\|^2_{L^2(\Omega)} ; \\
I_9 & \le C(\sigma) \left\| A^{j+1} \right\|^2_{L^2(\Omega)} \left\| \Delta \! m^{j+1} \right\|^2_{L^2(\Omega)} + C(\sigma) \left\| A^{j+1} \right\|^4_{L^2(\Omega)} + \sigma \left\| \Delta A^{j+1} \right\|^2_{L^2(\Omega)}.
\end{align*}
\]
\[ I_7, I_8 \leq C(\sigma) \left\| A^{j+1} \right\|^2_{L^2(\Omega)} + \left\| u^{j+1} \right\|^2_{H^1(\Omega)} + \sigma \left\| \Delta A^{j+1} \right\|^2_{L^2(\Omega)}. \]

Choose \( \sigma \) small enough, multiply with \( k \), add from \( 0 \) to \( l \) and get with the help of the discrete Gronwall lemma and Corollary 5.6 for \( k \) sufficiently small
\[
\max_{0 \leq j \leq J-1} \left\| \Delta m^{j+1} \right\|^2_{L^2(\Omega)} + k \sum_{j=0}^{J-1} \left\| \nabla \Delta m^{j+1} \right\|^2_{L^2(\Omega)} \leq C \left( K, \| m_0 \|_{H^2(\Omega)} , T \right).
\]

**Step 3:** Multiply (5.6) with \( dt \) \( A^{j+1} \), integrate in space and we get similar terms as in Step 1, where now the second entry is replaced by \( dt A^{j+1} \). There is just one new term which may be bounded by
\[
( m^{j+1} \times \Delta A^{j+1}, dt A^{j+1} ) \leq C(\sigma) \left\| m^{j+1} \right\|^2_{L^\infty(\Omega)} \left\| \nabla \Delta m^{j+1} \right\|^2_{L^2(\Omega)} + \sigma \left\| dt A^{j+1} \right\|^2_{L^2(\Omega)},
\]
for \( \sigma > 0 \). Choose \( \sigma \) small enough and get with Step 2 and Corollary 5.6 for \( k \) small
\[
k \sum_{j=0}^{J-1} \left\| dt \nabla m^{j+1} \right\|^2_{L^2(\Omega)} \leq C \left( K, \| m_0 \|_{H^2(\Omega)} , T \right).
\]

\[\square\]

6. The semi-discrete optimization problem

We consider for a fixed time step \( k \) the following semi-discrete optimization Problem 6.1, which is related to Problem 4.1. The analysis of this problem follows the steps in Section 4 and is meant to prepare the analysis in Section 7, where iterates of the corresponding semi-discrete optimality system (6.1) are shown to converge to a solution of (4.1).

We define the spaces
\[ M_k := \left[ H^2(\Omega) \right]^{J+1}, \quad \quad U_k := \left[ H^1(\Omega) \right]^J, \]
and use the notation
\[ M := \left( m^j \right)_{j=0}^J \in M_k, \quad U := \left( u^j \right)_{j=1}^J \in U_k \]
as well as the functional \( F_k : M_k \times U_k \rightarrow \mathbb{R}, \)
\[ F_k(M, U) := \frac{k}{2} \sum_{j=1}^J \left\| m^j - \bar{m}(t_j) \right\|^2_{L^2(\Omega)} + \frac{\lambda_0 k}{2} \sum_{j=1}^J \left\| u^j \right\|^2_{L^2(\Omega)} + \frac{\lambda_1 k}{2} \sum_{j=1}^J \left\| \nabla u^j \right\|^2_{L^2(\Omega)}. \]

**Problem 6.1.** Let \( T > 0 \), \( \bar{m} : [0, T] \times \Omega \rightarrow S^2 \) be a given smooth function, as well as \( m_0 \in H^2(\Omega) \) with \( |m_0|^2 = 1 \) in \( \Omega \), and \( \lambda_0, \lambda_1 > 0 \). For \( k > 0 \) fixed, find functions \( M^* \in M_k \) and \( U^* \in U_k \), such that
\[ (M^*, U^*) = \arg\min_{(M, U) \in M_k \times U_k} F_k(M, U) \text{ subject to (5.1)}. \]
6.1. Existence.

**Theorem 6.2.** There is a constant $k_0 \left( \| \vec{m} \|_{L^\infty(\Omega)}, \| m_0 \|_{H^2(\Omega)}, T \right) > 0$, such that for every $k \leq k_0$ there exists at least one minimum $(M^*, U^*) \in M_k \times U_k$ of Problem 6.1.

**Proof.** The proof is similar to the proof of its continuous counterpart, Theorem 4.2. Use the stability of the iterates, see Corollary 5.6 to get a minimizing sequence which is uniformly bounded in $k$. □

**Remark 6.3** (Stability of the optimal states and controls). Let $(\{M^*_k, U^*_k\})_{k>0}$ be solutions of Problem 6.1 for different meshes $(I_k)_{k>0}$. By Corollary 5.6 there exists for $k$ sufficiently small a constant $C > 0$ independent of $k$, such that for optimal controls $U^*$

$$k \sum_{j=1}^J \| u_j^* \|^2_{H^1(\Omega)} \leq C \inf_{(M, U) \in M_k \times U_k} \sup_{0 \leq j \leq J-1} \| m_j^* \|^2_{H^2(\Omega)} + k \sum_{j=0}^{J-1} \| m_j^* \|^2_{H^2(\Omega)} + k \sum_{j=0}^{J-1} \| d_j m_j^* \|^2_{H^2(\Omega)}$$

Furthermore, there exists a constant $C > 0$ independent of $k$, such that for the optimal states $M^*$ and $k > 0$ small enough by Lemma 5.8

$$\max_{0 \leq j \leq J} \| m_j^* \|_{H^2(\Omega)} \leq C \left( \| \vec{m} \|_{L^\infty(\Omega)}, \| m_0 \|_{H^2(\Omega)}, T \right).$$

6.2. Semi-discrete optimality system. We deduce the necessary optimality system for a minimum of Problem 6.1, hence we rephrase the semi-discrete problem to use the Lagrange multiplier theorem, cf. Subsection 4.2. To do this, we define the two functions $e_k : M_k \times U_k \rightarrow L^2(\Omega)^J$ and $a_k : M_k \times U_k \rightarrow H^2(\Omega)$ by

$$e_k(M, U) := \left( d_j m^j - \alpha \Delta m^j - \alpha |\nabla m^{j-1}|^2 m^j + \alpha m^j \times (m^{j-1} \times u^j) \right)^J_{j=1},$$

$$a_k(M, U) := m^0 - m_0.$$

The following problem is a reformulation of Problem 6.1

**Problem 6.4.** Let $T > 0$, $\vec{m} : [0, T] \times \Omega \rightarrow S^2$ be smooth and given, as well as $m_0 \in H^2(\Omega)$ with $|m_0|^2 = 1$ in $\Omega$, and $\lambda_0, \lambda_1 > 0$. Minimize for fixed $k > 0$ the functional $F_k$ subject to $H_k(M, U) = 0$, where

$$H_k : M_k \times U_k \rightarrow \left[ L^2(\Omega) \right]^J \times H^2(\Omega), \quad H_k(M, U) := \left( e_k(M, U) \right)^T_{a_k(M, U)}.$$

We check the assumptions needed to apply the Lagrange multiplier theorem; cf. [16] Chapter 9, Theorem 1.

**Lemma 6.5.** The function $H_k$ is continuously Fréchet differentiable, with derivative

$$\langle H'_k(M, U), (\delta M, \delta U) \rangle = \left( \langle e'_k(M, U), (\delta M, \delta U) \rangle \right)^T_{\left( \langle a'_k(M, U), (\delta M, \delta U) \rangle \right)}.$$
where
\[
\langle e_k(M, U), (\delta M, \delta U) \rangle = \left( d_i \delta m^j - \alpha \Delta \delta m^j - \alpha |\nabla m^{j-1}|^2 \delta m^j + \alpha \delta m^j \times (m^{j-1} \times u^j) - m^j \times \Delta \delta m^j - \delta m^j \times \Delta m^j - \delta m^j \times u^j - 2\alpha (\nabla m^{j-1}, \nabla \delta m^j)m^j + \alpha m^j \times (\delta m^{j-1} \times u^j) + \alpha m^j \times (m^{j-1} \times \delta u^j) - m^j \times \delta u^j \right)_{j=1},
\]
\[
\langle a_k(M, U), (\delta M, \delta U) \rangle = \delta m^0.
\]

Now we show that minima of Problem 6.1 are regular points.

**Lemma 6.6.** Let \((M^*, U^*)\) be a minimum of Problem 6.1. There exists a constant \(k_0 \left( \|\tilde{m}\|_{L^\infty(L^\infty)}, \|m_0\|_{H^2(\Omega)}, T \right) > 0\), such that \((M^*, U^*)\) is a regular point of \(H_k\) for all \(k \leq k_0\).

**Proof.** It is sufficient to show that for a fixed \(U \in U_k\) (e.g. for \(U = 0\)), the mapping
\[
(V, 0) \mapsto \langle H_k^*(M^*, U^*), (V, 0) \rangle : M_k \times U_k \rightarrow \left[ L^2(\Omega) \right]^J \times H^2(\Omega)
\]
is surjective. Using the fact that minima \(M^*\) are bounded independently from \(k\), together with the help of Remark 6.3 we get the surjectivity from the Lax-Milgram lemma for \(k_0\) small enough. 

It is due to Lemmas 6.5 and 6.6 that all assumptions of the Lagrange multiplier theorem, [16] Chapter 9, Theorem 1, are fulfilled; as a consequence there is a constant \(k_0 = k_0 \left( \|\tilde{m}\|_{L^\infty(L^\infty)}, \|m_0\|_{H^2(\Omega)}, T \right) > 0\) such that for all \(k \leq k_0\) there exists \((Z_k, \zeta_k) \in \left[ L^2(\Omega) \right]^J \times H^2(\Omega)^*\), such that the Lagrange functional \(L_k : M_k \times U_k \rightarrow \mathbb{R}\),

\[
L_k(M, U) := F_k(M, U) + \langle Z_k, e_k(M, U) \rangle + \langle \zeta_k, a_k(M, U) \rangle,
\]
is stationary at a minimum \((M^*, U^*)\), hence

\[
DL_k(M^*, U^*) = DF_k(M^*, U^*) + \langle Z_k, De_k(M^*, U^*) \rangle + \langle \zeta_k, Da_k(M^*, U^*) \rangle
\]

\[
= DF_k(M^*, U^*) + k \sum_{i=0}^{J-1} \langle \tilde{z}^i, De_k(M^*, U^*) \rangle + \langle \zeta_k, Da_k(M^*, U^*) \rangle = 0,
\]

where \(k\tilde{z}^i = z^i\). To simplify notation, we write \(z^i\) for \(\tilde{z}^i\).

We now compute the directional derivatives where all entries vanish except for the \(j\)-th component; see Lemma 6.3. If combined with the state equation, the necessary optimality system for a minimum \((M^*, U^*)\) of Problem 6.1 reads for \(k\) small enough:

\[
0 = d_i m^j_i - \alpha \Delta m^j_i - \alpha |\nabla m^{j-1}|^2 m^j_i + \alpha m^j_i \times (m^{j-1} \times u^j_i) - m^j_i \times \Delta m^j_i,
\]
\[
0 = \lambda_0 u^j_i - \lambda_1 \Delta u^j_i + \alpha (z^{j-1} \times m^j_i) \times m^{j-1}_i - z^{j-1} \times m^j_i,
\]
\[
0 = (m^j_i - \tilde{m}(t_j), \delta m) - (d_i z^j, \delta m) - \alpha (z^{j-1}, \Delta \delta m) - \alpha (z^{j-1}, |\nabla m^{j-1}|^2 \delta m) + \alpha (z^{j-1}, \delta m \times (m^{j-1} \times u^j_i)) - (z^{j-1}, m^j_i \times \delta m).
\]
We control arising terms separately. Let parts to conclude

\[ \text{Step 1:} \]

The pair

\[ \text{Proof.} \]

\[ C \]

where

\[ \text{Lemma 6.8.} \] Let

\[ \text{Remark 6.7 (Lagrange multiplier } \zeta_k). \] The derivation of \( L_k \) in direction \( m_0 \) leads to

\[ k \left( z^0, -\frac{1}{k} \delta m^0 - 2\alpha (\nabla m^0, \nabla \delta m^0) m^1 + \alpha k (z^0, m^1_k \times (\delta m^0 \times u^1_i)) + (\zeta_k, \delta m^0) = 0 \]

for all \( \delta m^0 \in H^2(\Omega) \), and \( \zeta_k \) is calculated by the iterates of the Lagrange multiplier \( Z_k \). Like in the continuous case, this property is not needed in the following.

6.3. Stability of the semi-discrete adjoint and optimal control. The Lagrange multiplier theorem yields that \( Z_k \in \left[ L^2(\Omega) \right]^j \) exists; however, improved stability properties are valid for the semi-discrete adjoint and for the optimal control, which we will need in Section 7 to pass to the limit for the semi-discrete optimality system (6.1).

**Lemma 6.8.** Let \((M^*, U^*)\) be a solution of Problem 6.1. There exists a constant \( k_0 \left( \| \bar{m} \|_{L^\infty(L^\infty)}, \| m_0 \|_{H^2(\Omega)}, T \right) > 0 \), such that for all \( k \leq k_0 \) the solution \( \{z^j\}^J_{j=0} \) of the semi-discrete adjoint equation (6.1c) with \( z^J = 0 \) satisfies the bounds

\[ \max_{0 \leq j \leq J} \| z^j \|_{H^1(\Omega)}^2 + k \sum_{j=1}^J \| dz^j \|_{L^2(\Omega)}^2 + k \sum_{j=0}^J \| z^j \|_{L^2(\Omega)}^2 \leq C \left( \| \bar{m} \|_{L^\infty(L^\infty)}, \| m_0 \|_{H^2(\Omega)}, T \right), \]

where \( C \) is independent of the step size \( k \).

**Proof.** The pair \((M^*, U^*)\) is a solution of Problem 6.1 and hence satisfies the stability properties in Remark 6.3.

**Step 1:** Test (6.1c) with \( z^{j-1} \) and use properties of the vector product and integration by parts to conclude

\[ -\frac{1}{2} \| z^{j-1} \|_{L^2(\Omega)}^2 + \alpha \| \nabla z^{j-1} \|_{L^2(\Omega)}^2 \leq \left( \bar{m}(t_j) - m_j^i, z^{j-1} \right) + \alpha \left( z^{j-1}, |\nabla m_j^i|^{2} z^{j-1} \right) + \left( z^{j-1}, \nabla m_j^i \times \nabla z^{j-1} \right) + 2\alpha \left( z^j, \nabla z^j, \nabla z^j, m_j^{i+1} \right) - \alpha \left( z^j, m_j^{i+1} \times (z^{j-1} \times u^j_i) \right) \]

\[ =: I_1 + \alpha I_2 + I_3 + 2\alpha I_4 + \alpha I_5. \]

We control arising terms separately. Let \( \sigma > 0 \).

\[ I_1 \leq \frac{1}{2} \| \bar{m}(t_j) - m_j^i \|_{L^2(\Omega)}^2 + \frac{1}{2} \| z^{j-1} \|_{L^2(\Omega)}^2, \]
We bound the terms separately. Let
\[ I_2 \leq \left\| \nabla m^{j-1}_{t} \right\|_{L^\infty(\Omega)} \frac{2}{\Delta t} \left\| z^{j-1} \right\|_{L^2(\Omega)}, \]
\[ I_3 \leq C(\sigma) \left\| \nabla m^{j}_{t} \right\|_{L^\infty(\Omega)} \left\| z^{j-1} \right\|_{L^2(\Omega)} + \sigma \left\| \nabla z^{j-1} \right\|_{L^2(\Omega)}, \]
\[ I_4 \leq C(\sigma) \left\| \nabla m^{j}_{t} \right\|_{L^\infty(\Omega)} \left\| m^{j+1}_{t} \right\|_{L^\infty(\Omega)} + \sigma \left\| z^{j} \right\|_{L^2(\Omega)}, \]
\[ I_5 \leq C \left\| m^{j+1}_{t} \right\|_{L^\infty(\Omega)} \left\| u^{j+1}_{t} \right\|_{L^\infty(\Omega)} \left\| z^{j} \right\|_{L^2(\Omega)} + C \left\| z^{j-1} \right\|_{L^2(\Omega)}. \]

Choose \( \sigma \) small enough, multiply by \( k \), sum up and get with the discrete Gronwall lemma and Remark 6.3
\[
\max_{1 \leq j \leq J} \left\| z^{j-1} \right\|_{L^2(\Omega)} + k \sum_{j=1}^{J} \left\| \nabla z^{j-1} \right\|_{L^2(\Omega)} \leq C \left( \left\| \bar{m} \right\|_{L^\infty(\Omega) \cap L^2(\Omega)}, \left\| m_0 \right\|_{H^2(\Omega)}, T \right).
\]

**Step 2:** We proceed formally, and test \([6.1b] \) with \( -\Delta z^{j-1} \). We use integration by parts and the properties of the vector product to infer the estimate
\[
- \frac{1}{2} \partial_t \left\| \nabla z^{j-1} \right\|_{L^2(\Omega)} + \alpha \left\| \Delta z^{j-1} \right\|_{L^2(\Omega)} \leq - \left( \bar{m}(t_j) - m^{j}_s, \Delta z^{j-1} \right) - \alpha \left( z^{j-1}, \left\| \nabla m^{j-1}_{t} \right\|_{L^2(\Omega)} \right) - 2 \left( \nabla z^{j-1}, \nabla m^{j}_{t} \times \Delta z^{j-1} \right) + \left( z^{j-1}, \Delta m^{j}_{t} \times \Delta z^{j-1} \right) - \alpha \left( z^{j-1}, \Delta z^{j-1} \times m^{j}_{t} \right) - \alpha \frac{\left( \nabla z^{j}, \nabla m^{j}_{t}, \Delta z^{j-1} \right) m^{j+1}_{t}}{\left( \nabla z^{j}, \Delta m^{j}_{t}, \Delta z^{j-1} \right) m^{j+1}_{t}} + \left( z^{j}, \Delta z^{j-1} \times u^{j+1}_{t} \right) =: I_1 + \alpha I_2 + I_3 + I_4 + \alpha I_5 + I_6 + I_7 + 2\alpha (I_8 + \ldots + I_{10}) + I_{11}.
\]

We bound the terms separately. Let \( \sigma > 0 \).
\[ I_1 \leq C(\sigma) \left\| \bar{m}(t_j) - m^{j}_s \right\|_{L^2(\Omega)} + \sigma \left\| \Delta z^{j-1} \right\|_{L^2(\Omega)} ,
\]
\[ I_2 \leq C(\sigma) \left\| z^{j-1} \right\|_{L^2(\Omega)} + \sigma \left\| \Delta z^{j-1} \right\|_{L^2(\Omega)},
\]
\[ I_3 \leq C(\sigma) \left\| \nabla m^{j}_s \right\|_{L^\infty(\Omega)} \left\| \nabla z^{j-1} \right\|_{L^2(\Omega)} + \sigma \left\| \Delta z^{j-1} \right\|_{L^2(\Omega)},
\]
\[ I_4 + I_6 = 0,
\]
\[ I_5 \leq C(\sigma) \left\| z^{j-1} \right\|_{L^2(\Omega)} + \sigma \left\| \Delta z^{j-1} \right\|_{L^2(\Omega)},
\]
\[ I_7 \leq C(\sigma) \left\| z^{j-1} \right\|_{L^2(\Omega)} + \sigma \left\| \Delta z^{j-1} \right\|_{L^2(\Omega)},
\]
\[ I_8 \leq C(\sigma) \left\| \nabla m^{j}_s \right\|_{L^\infty(\Omega)} \left\| \nabla z^{j-1} \right\|_{L^2(\Omega)} + \sigma \left\| \Delta z^{j-1} \right\|_{L^2(\Omega)},
\]
\[ I_9 \leq C(\sigma) \left\| \Delta m^{j}_s \right\|_{L^2(\Omega)} \left\| \nabla z^{j-1} \right\|_{L^2(\Omega)} + \sigma \left\| \Delta z^{j-1} \right\|_{L^2(\Omega)},
\]
\[ I_{10} \leq C(\sigma) \left\| \nabla m^{j}_s \right\|_{L^\infty(\Omega)} \left\| \nabla z^{j-1} \right\|_{L^2(\Omega)} + \sigma \left\| \Delta z^{j-1} \right\|_{L^2(\Omega)}.
\]
Proposition 6.9. Let $\lambda$ similar to Lemma 4.7. Now we are able to show an improved stability property for an optimal control $u^*$.

**Proof.**

Choose $\sigma$ small enough, multiply with $k$, sum up, and use Step 1 and Remark 6.3 to get

$$\max_{0 \leq j \leq J} \|\nabla z_j^1\|_L^2(\Omega)^2 + \frac{\alpha k}{2} \sum_{j=1}^J \|\Delta z_j^1\|_L^2(\Omega)^2 \leq C \left( \|\bar{m}\|_{L^\infty(\Omega)}, \|m_0\|_{H^2(\Omega)}, T \right).$$

**Step 3:** Test (6.1c) with $-d_t z_j$ and get via integration by parts similar terms as in Step 2, where now the second entry is replaced by $d_t z_j$. There is just one additional term, which we can bound by

$$(\Delta z_j^1, m^j \times d_t z_j) \leq C(\sigma) \|\Delta z_j^1\|^2_{L^2(\Omega)} \|m^j\|^2_{L^\infty(\Omega)} + \sigma \|d_t z_j\|^2_{L^2(\Omega)},$$

for $\sigma > 0$. Choose $\sigma$ small enough and get with Remark 6.3 and Step 1 and 2, respectively,

$$k \sum_{j=1}^J \|d_t z_j\|^2_{L^2(\Omega)} \leq C \left( \|\bar{m}\|_{L^\infty(\Omega)}, \|m_0\|_{H^2(\Omega)}, T \right).$$

Now we are able to show an improved stability property for an optimal control $U^*$ of Problem 6.1 similar to Lemma 4.7.

**Proposition 6.9.** Let $(M^*, U^*)$ be a solution of Problem 6.1. There exists a constant $k_0 \left( \|\bar{m}\|_{L^\infty(\Omega)}, \|m_0\|_{H^2(\Omega)}, T \right) > 0$, such that for all $k \leq k_0$ the optimal control $U^* \in H^2(\Omega)^J$ satisfies the bounds

$$\max_{1 \leq j \leq J} \|u^j\|^2_{H^2(\Omega)} + k \sum_{j=1}^J \|d_t u^j\|^2_{H^1(\Omega)} \leq C \left( \|\bar{m}\|_{L^\infty(\Omega)}, \|m_0\|_{H^2(\Omega)}, T \right),$$

where $u^0 := u^*$ and $C$ is independent of the time step $k$.

**Proof.**

**Step 1:** Multiply (6.1b) with $u^j$ and $-\Delta u^j$, respectively, and use Lemma 6.8 and Remark 6.3 to get

$$\max_{1 \leq j \leq J} \|u^j\|^2_{L^2(\Omega)} + \max_{1 \leq j \leq J} \|\nabla u^j\|^2_{L^2(\Omega)} + \max_{1 \leq j \leq J} \|\Delta u^j\|^2_{L^2(\Omega)} \leq C \left( \|\bar{m}\|_{L^\infty(\Omega)}, \|m_0\|_{H^2(\Omega)}, T \right).$$

**Step 2:** Take the discrete time derivative of (6.1b) and get for $j = 2, \ldots, J$

$$\lambda_0 d_t u^j - \lambda_1 d_t \Delta u^j = -\alpha (d_t z_j - 1 \times m^j) \times m^{j-1} - \frac{1}{k} (z^{j-2} \times d_t m^j) \times m^{j-1}$$

$$- \frac{1}{k} (z^{j-2} \times m^{j-1}) \times d_t m^j + d_t z_j - 1 \times m^j + z^{j-2} \times d_t m^j.$$
Lemma 7.1. Let \( k > 0 \), and \( \{(M, U)\}_{k \geq 0} \) be solutions of the Problem 6.1. There exist \( u^* \in L^\infty(H^2) \cap H^1(H^1), m^* \in L^2(H^3) \cap L^\infty(H^3) \cap H^1(H^1) \) and \( z^* \in L^2(H^2) \cap L^\infty(H^1) \cap H^1(L^2) \), such that a subsequence of the interpolants of \( M, U \) and \( Z \) converges in the following sense \((k \to 0)\),

\[
\begin{align*}
\mathcal{U}^+, \mathcal{U}^+, \mathcal{U}^* &\to u^* &\text{strongly in } L^2(H^1), \\
\mathcal{U}^+, \mathcal{U}^* &\to u^* &\text{weakly in } L^2(H^2), \\
\frac{d}{dt} \mathcal{U} &\to \frac{d}{dt} u^* &\text{weakly in } L^2(H^1), \\
\mathcal{M}^+, \mathcal{M}^-, \mathcal{M}^*, \mathcal{M} &\to m^* &\text{weakly-star in } L^\infty(H^2), \\
\mathcal{M}^+, \mathcal{M}^-, \mathcal{M}^*, \mathcal{M} &\to m^* &\text{weakly in } L^2(H^3), \\
\frac{d}{dt} \mathcal{M} &\to \frac{d}{dt} m^* &\text{weakly in } L^2(H^1), \\
\mathcal{M}^+, \mathcal{M}^-, \mathcal{M} &\to m^* &\text{weakly-star in } L^\infty(H^2), \\
Z^+, Z &\to z^* &\text{strongly in } L^2(L^2), \\
\frac{d}{dt} Z &\to \frac{d}{dt} z^* &\text{weakly in } L^2(H^2), \\
Z^+, Z &\to z^* &\text{weakly in } L^2(L^2), \\
\frac{d}{dt} Z &\to \frac{d}{dt} z^* &\text{weakly-star in } L^\infty(H^1).
\end{align*}
\]

Here we set \( u^0 := u^1 \) for the construction of the in time interpolant \( U \).

Proof. The weak and weak-star convergences follow from the uniform bounds of the interpolants in Remark 6.3, Lemma 6.8, and Proposition 6.9. The strong convergence properties follow accordingly with the help of the Aubin-Lions’ theorem for the affine time interpolants. The uniform bounds of the time derivatives ensure that the limits of the affine interpolants coincide with those of the piecewise constant interpolants.

It now remains to show that \((m^*, u^*, z^*)\) solves the continuous optimality system (4.1). The convergence properties stated in Lemma 7.1 are sufficient to verify that all linear terms in 6.1 converge to their continuous counterparts in (4.1), so we only consider the nonlinear terms in the following lemma.

Lemma 7.2. Let \((m^*, u^*, z^*)\) be from Lemma 7.1 as well as the convergent subsequence \(\{(M, U, Z)\}_{k \geq 0}\). For all \( \varphi \in C^\infty(C^\infty) \), we have for \( k \to 0 \)

\[
\int_0^T (|\nabla \mathcal{M}^-|^2 \mathcal{M}^+, \varphi) \, ds - \int_0^T (|\nabla m^*|^2 m^*, \varphi) \, ds \to 0,
\]

\( \star \)
\begin{align}
&\int_0^T (\mathcal{M}^+ \times (\mathcal{M}^- \times \mathcal{U}^+), \varphi) \, ds - \int_0^T (m^* \times (m^* \times u^*), \varphi) \, ds \to 0, \\
&\int_0^T (\mathcal{M}^+ \times \Delta \mathcal{M}^+, \varphi) \, ds - \int_0^T (m^* \times \Delta m^*, \varphi) \, ds \to 0, \\
&\int_0^T (\mathcal{M}^+ \times \mathcal{U}^+, \varphi) \, ds - \int_0^T (m^* \times u^*, \varphi) \, ds \to 0, \\
&\int_0^T ((z^- \times m^+) \times \mathcal{M}^-, \varphi) \, ds - \int_0^T ((z^* \times m^+) \times m^*, \varphi) \, ds \to 0, \\
&\int_0^T (\mathcal{Z}^- \times \mathcal{M}^+, \varphi) \, ds - \int_0^T (z^* \times m^*, \varphi) \, ds \to 0, \\
&\int_0^T (\mathcal{Z}^+, \varphi \times (m^* \times u^*)) \, ds - \int_0^T (z^*, \varphi \times (m^* \times u^*)) \, ds \to 0, \\
&\int_0^T (\mathcal{Z}^-, \varphi \times (\mathcal{M}^- \times \mathcal{U}^+)) \, ds - \int_0^T (z^*, \varphi \times (\mathcal{M}^- \times \mathcal{U}^+)) \, ds \to 0, \\
&\int_0^T (\mathcal{Z}^-, \mathcal{M}^+ \times \Delta \varphi) \, ds - \int_0^T (z^*, m^* \times \Delta \varphi) \, ds \to 0, \\
&\int_0^T (\mathcal{Z}^-, \varphi \times \mathcal{U}^+) \, ds - \int_0^T (z^*, \varphi \times \mathcal{U}^+) \, ds \to 0, \\
&\int_0^T (\mathcal{Z}^+, \varphi \times \mathcal{M}^+) \, ds - \int_0^T (z^*, \varphi \times \mathcal{M}^+) \, ds \to 0, \\
&\int_0^T (\mathcal{Z}^+, \varphi \times (\varphi \times \mathcal{U}^*)) \, ds - \int_0^T (z^*, m^* \times (\varphi \times \mathcal{U}^*)) \, ds \to 0.
\end{align}

**Proof.** For each term it is not difficult to pass to the limit, using smooth test functions $\varphi \in \mathcal{C}^\infty(\mathcal{C}^\infty)$ and the convergence properties of the iterates in Lemma 7.1. A detailed discussion on each term can also be found in [21, Section 5]. \hfill \Box

**Remark 7.3** (Initial and terminal condition). We use integration by parts, and $\varphi \in \mathcal{C}^\infty(\mathcal{C}^\infty)$ such that $\varphi(T) = 0$ to conclude for $k \to 0$ that

$$
\int_0^T (m^*_t, \varphi) \, ds \leftarrow \int_0^T (\mathcal{M}_t, \varphi) \, ds
\right.
$$

$$
= -\int_0^T (\mathcal{M}, \varphi_t) \, ds - (m_0, \varphi(0)) \to -\int_0^T (m^*, \varphi_t) \, ds - (m_0, \varphi(0)).
\right.
$$

Consequently, we have $m^*(0) = m_0$. For the adjoint $z^*$ we get $z^*(T) = 0$ analogously.

We may now state the main result of this work, which asserts the construction of solutions of the optimality system (4.1) by proper limits of those from (6.1) for $k \to 0$. Moreover, we have a strong convergence of the optimal controls (up to a subsequence), whereof we are interested if we want to influence our ferromagnet in an optimal way.

**Theorem 7.4.** There exist functions $u^* \in L^\infty(\mathcal{H}^2) \cap H^1(\mathcal{H}^1)$, $m^* \in L^2(\mathcal{H}^3) \cap L^\infty(\mathcal{H}^2) \cap H^1(\mathcal{H}^1)$ and $z^* \in L^2(\mathcal{H}^2) \cap L^\infty(\mathcal{H}^1) \cap H^1(\mathcal{L}^2)$, which are defined in Lemma 7.1, such that solutions of the semi-discrete optimality system (6.1) converge to them in proper norms (up to subsequences) for $k \to 0$, and these limits solve the continuous optimality system (4.1).

In addition, there exists a subsequence of the semi-discrete optimal controls $\{U\}$, such that $U^+, U \to u^*$ strongly in $L^2(\mathcal{H}^1)$ for $k \to 0$.

**Proof.** From the first four assertions in Lemma 7.2 we get that a subsequence of solutions of (6.1a) converges to a solution of (4.1a). The next two assertions there establish the convergence of a subsequence of solutions of (6.1b) to a solution of (4.1b) and with the remaining assertions of Lemma 7.2 we deduce that a subsequence of solutions of (6.1c) converges to a solution of (4.1c). Hence $(m^*, u^*, z^*)$ fulfills the continuous optimality system (4.1).

The strong convergence of a subsequence of the optimal controls follows by Lemma 7.1. \hfill \Box
In this section we compare two numerical schemes to approximate the necessary optimality system \((4.1)\): The first is the discrete optimality system which is corresponding to the semi-discrete optimality system \((6.1)\) and bases on a natural finite element formulation of \((5.1)\). The second discrete optimality system uses a projection idea for the semi-discretized state equation, cf. \((8.2)\), and for the full discretization we use again a natural finite element formulation. Well-posedness and convergence analysis of the projection method in the case of absent control has been studied in [18, Chapter 4] for sufficiently small time steps \(k \leq k_0(T)\), where iterates are also shown to better approximate the target manifold \(S^2\) than those of \((5.1)\).

8.1. Semi-discrete optimality system for a projection scheme. We replace the semi-discrete state equation \((5.1)\) by the following projection scheme:

\[
\text{Scheme 8.1. Let } m^0 := m_0. \text{ For } j \geq 0:\n\]

1. Compute \(m^j : \Omega \to S^2\) according to \(m^j = \frac{m^j}{|m^j|}\). (8.1)

2. Let \(m^{j+1} : \Omega \to \mathbb{R}^3\) be the solution of
\[
\frac{1}{k} (m^{j+1} - m^j) - \alpha \Delta m^{j+1} = \alpha |\nabla m^j|^2 m^{j+1} - \alpha m^{j+1} \times (m^j \times u^{j+1}) + m^{j+1} \times \Delta m^{j+1} + m^{j+1} \times u^{j+1}. \quad (8.2)
\]

Inserting (8.1) into (8.2) allows to recast the leading term in the following form,
\[
\frac{1}{k} (m^{j+1} - m^j) = d_j m^{j+1} + \frac{1}{k} \left( 1 - \frac{1}{|m^j|} \right) m^j.
\]

As a consequence, the projection method may be interpreted as a semi-explicit penalization method, with penalization parameter \(\varepsilon = k\).

Next, we turn to the corresponding semi-discrete optimization problem.

**Problem 8.2.** Let \(T > 0\), \(m : [0, T] \times \Omega \to S^2\) be smooth and given, as well as \(m_0 \in H^2(\Omega)\) with \(|m_0|^2 = 1\) in \(\Omega\), and \(\lambda_0, \lambda_1 > 0\). For \(k > 0\) fixed, find functions \(M^* \in M_k\) and \(U^* \in U_k\), such that
\[
(M^*, U^*) = \argmin_{(M, U) \in M_k \times U_k} F_k(M, U) \quad \text{subject to (8.2)}.
\]

The corresponding necessary optimality system for a minimum \((M^*, U^*)\) of Problem 8.2 reads as follows: For all \(j = 1, \ldots, J\),
\[
0 = d_j m_*^j - \alpha \Delta m_*^j - \alpha |\nabla m_*^{j-1}|^2 m_*^j + \alpha m_*^j \times (m_*^{j-1} \times u_*^j) - m_*^j \times \Delta m_*^j - m_*^j \times u_*^j + \frac{1}{k} \left( 1 - \frac{1}{|m_*^{j-1}|} \right) m_*^{j-1}, \quad (8.3a)
\]
0 = \lambda_0 u^j + \lambda_1 \Delta u^j + \alpha(z_j \times m^j) \times m^j - z_j - x_j \times m^j,
0 = (m^j - \bar{m}(t_j), \delta m) - (d \delta z, \delta m) - \alpha(z_j, \Delta \delta m) - \alpha(|\nabla m^j| \delta m)
+ \alpha(z_j, m^j \times \Delta \delta m) - \alpha(z_j, m^j \times \Delta \delta m) + \alpha(z_j, m^j+1 \times (\delta m \times u^j+1))
+ \frac{1}{k} \left( 1 - \frac{1}{m^j} \right) \delta m, z_j + \frac{2}{k} \left( \frac{1}{|m^j|} \delta m, m^j \right)
\quad \text{for all } \delta m \in H^2(\Omega),
\quad \text{together with the initial condition } m^0 = m_0, \quad \text{and terminal condition } z^j = 0, \quad \text{respectively.}

We compare the above system with (6.1): we already discussed the (additional) penalization term which arises in (8.3a). The optimality condition (8.3b) has not changed. The new last two terms in the adjoint equation (8.3c) are attributed to the projection strategy for the discrete state equation. These modifications create new challenges for a corresponding stability and convergence analysis for system (8.3), respectively, which we leave open for further research. Interestingly enough, the computational studies which are reported below indicate unexpected effects for this system, such as an additional (approximate) orthogonality of the optimal state and the adjoint variables – as a consequence of the enhanced approximation of the target manifold \( \mathbb{S}^2 \); cf. Table 1 and Figure 4 below.

We note that the approximate orthogonality between \( u^j \) and \( m^j \) or \( m^j \) follows from (8.3b), while the orthogonality of \( m^j \) and \( z^j \) is indicated by the following formal calculation: Inserting \( \delta m = \langle m^j, z^j \rangle m^j \) into (8.3c) leads with the help of the Grassmann identity to a positive term

\[ \frac{2}{k} \left\| \frac{1}{m^j} \langle m^j, z^j \rangle \right\|_{L^2(\Omega)}^2. \] (8.4)

If all other terms are bounded when choosing this test function, we see that the term in (8.4) is uniformly bounded with respect to \( k \) after summation, i.e., \( m^j \) and \( z^j \) are orthogonal in the limit for \( k \to 0 \). An issue to overcome for this purpose is to e.g. bound the last but one term in (8.3c) uniformly with respect to \( k \), which requires a sharp estimate for \( |1 - \frac{1}{m^j}| \) pointwise in space and time. Also it is not clear if (8.3c) converges to the proper limit (4.1c) since the two additional terms in the last line of (8.3c) have to converge to zero. For the first term, this problem could be solved if one could prove that \( |1 - \frac{1}{m^j}| \) converges to zero superlinearly in \( k \); but the second term is only bounded by the argument given above. A possible solution could be to use the fact that the last term in (8.3c) is parallel to \( m^j \), cf. [9]. We can take the cross product of (8.3c) with \( m^j \), which causes the term to vanish, and then pass to the limit in the equation \( m^j \times (8.3c) \) with the good regularity properties for state \( m^j \) and adjoint \( z^j \). The limit is then parallel to \( m^j \), i.e., we get an additional term \( \mu m^j \) into the limit equation, and it has to be calculated if the coefficient \( \mu \) is correct and the limit equation coincides with (1.1c). To conclude, a rigorous convergence analysis for system (8.3) which is motivated from the projection method in Scheme 8.1 requires further investigation and is left as an open problem.
8.2. Description of the simulations. Next, we consider the space-time discretization of both schemes [6.1] and [8.3]. Therefore, we choose $V_h$-valued controls $u^j_h$, states $m^j_h$ and adjoint variables $z^j_h$, where $V_h$ denotes the space of vector-valued continuous, piecewise polynomial functions of maximum degree one. By $U_h = (u^j_h)_{j=1}^J$, $M_h = (m^j_h)_{j=0}^J$ and $Z_h = (z^j_h)_{j=0}^J$, we denote the collection of the resulting finite element functions.

The corresponding nonlinear system of equations is solved by using a discrete version of Newton’s method. There, the maximum amount of iterations is set to be 10,000, and the tolerance used is $\delta_{\text{Newton}} = 10^{-10}$. The required amount of iterations of the nonlinear algebraic solver for (8.3a) (average 2.16/max. 5 iterations) is comparable with (6.1a) (average 2.15/max. 5 iterations) for our considered standard choice of initial value, desired state and discretization parameters.

Lemma 5.2 asserts that the semi-discrete equation (5.1) has a unique solution and it seems that the same holds for a fully discrete solution of the state equation. This allows us to use a solution operator $M_h = M_h(U_h)$, and to restate the cost functional $F_k(U_h) := F_k(M_h(U_h), U_h)$. This is necessary in order to use a black box-type method like the steepest descent algorithm as indicated below. We refer to [12, 13] for the theoretical background, and a recent overview about algorithms used for solving the optimization problem.

In order to minimize the functional
\[
\tilde{F}_k(U_h) = \frac{k}{2} \sum_{j=1}^J \|m^j_h(u^j_h) - \bar{m}(t_j)\|^2_{L^2(\Omega)} + \frac{\lambda_0 k}{2} \sum_{j=1}^J \|u^j_h\|^2_{L^2(\Omega)} + \frac{\lambda_1 k}{2} \sum_{j=1}^J \|\nabla u^j_h\|^2_{L^2(\Omega)},
\]
a modified gradient descent algorithm is used, which is similar to the Barzilai and Borwein criterion applied in [7]. The corresponding algorithm reads as follows:

**Algorithm 8.3.** Set $U^0_h \equiv 0$ and fix the positive constants $\sigma_{\text{init}}$, $\sigma$, $\sigma^*$ and $\delta_{\text{tol}}$. Iterate for $r \geq 0$:

1. For $r \in \{0, 1\}$, set $\sigma_r = \sigma_{\text{init}}$.
2. For $r \geq 2$, compute:
   \[
   \sigma_r = \sigma^* \cdot \frac{\int_0^T \left(W_h^{r-1} - G_h^{r-1}\right) dt}{\|W_h^{r-1}\|^2_{L^2(\Omega)}},
   \]
   where $W_h^r := U_h^r - U_h^{r-1}$, and $G_h^r := \nabla_U \tilde{F}_k(U_h^r) - \nabla_U \tilde{F}_k(U_h^{r-1})$.
3. Check: If $\sigma_r \notin [\underline{\sigma}, \overline{\sigma}]$, then set $\sigma_r = \sigma_{\text{init}}$.
4. Compute
   \[
   U_h^{r+1} = U_h^r - \sigma_r \nabla_U \tilde{F}_k(U_h^r).
   \]
5. Stop the algorithm, if $\left\|\nabla_U \tilde{F}_k(U_h^r)\right\|^2_{L^2(\Omega)} \leq \delta_{\text{tol}}$.

In all the studies below, we choose constants $\sigma_{\text{init}} = 0.01$, $\sigma^* = 0.2$, $\underline{\sigma} = 0.01$ and $\overline{\sigma} = 0.1$. The stopping condition is set to be $\delta_{\text{tol}} = 0.00005$, which is obtained after 2,000 up to 50,000 iteration steps, strongly depending on the choice of the prefactors $\lambda_0, \lambda_1$. 

For all our studies, we consider the domain \( \Omega = (0, 1) \), and as initial value the function
\[
m_0(x) = \begin{cases} 
0 & \text{for } x \in [0, 0.5), \\
\sin(2\pi x) & \text{for } x \in [0.5, 1). 
\end{cases}
\]
Fix the damping parameter to be \( \alpha = 0.1 \). The time interval \([0, T]\) is set to be \([0, 1]\). If not pointed out otherwise, the fixed time-discretization parameter \( k = 0.002 \), the fixed spatial mesh size \( h = 0.05 \), and the fixed prefactor \( \lambda_1 = 10^{-4} \) are used. The desired magnetization profile \( \tilde{m}(t) \) is set to be \((0, 1, 0)^T \) for \( 0 \leq t < 0.5 \), which then switches to \((0, -1, 0)^T \) for \( 0.5 \leq t \leq 1 \); see Figure 3 (red).

Next, we perform two different simulations, the first allowing for large values of the control by setting \( \lambda_0 = 0.005 \) in the functional; in the second series of studies, we furtherly penalize the \( L^2(L^2) \)-norm of the control by choosing \( \lambda_0 = 0.02 \). For the first choice, convergence with rates and possible orthogonalities of the variables will be considered.

In the following, we refer to (6.1) as ‘SI scheme’, and to (8.3) as ‘Projected method’.

8.3. Simulations using \( \lambda_0 = 0.005 \).

8.3.1. Comparison of the computed solutions. An approximation of the solution of the necessary optimality system (4.1) via the SI scheme is shown in Figure 3. The desired profile \( \tilde{m} \) is visualized in red, the resulting optimal control \( U_h^* \) is blue, and the optimal state \( M_h^* \) is black.

The dynamics starts in a way that the control tries to force the optimal state to be close to the desired state; this effect goes along with large gradients in the applied control. At a bit later times (see e.g. the snapshot for \( t = 0.4 \)), the system anticipates the switching and starts the changeover, which is completed at times close to \( t = 0.7 \). This switching can also be observed by regarding the time evolution of the functional and its components, see Figure 2. The discrete dynamics which uses the Projected method differs only slightly: The fact that its iterates stay closer to the sphere (see Figure 4 (left)) yields a larger distance to the desired state during the switching, see again Figure 2.
Figure 2. Functional and components of the functional compared for the SI scheme and the Projected method.
Figure 3. Optimal pair $(M_h^*, U_h^*)$ (black, blue) simulated using the SI scheme.
8.3.2. Relative orientation of state, control and adjoint variable. Next, some numerical tests are performed to investigate the relative orientation of the state $M^*_h$, the control $U^*_h$, and the adjoint variable $Z_h$.

As already mentioned in the introduction, orthogonality of the optimal state $m^*$ and the optimal control $u^*$ holds for $\lambda_1 = 0$, while possible orthogonality of iterates of the semi-discrete problem fails to hold due to a time shift (see equation (1.7)). However, (approximate) orthogonality of $M^*_h$ and $U^*_h$ is expected to hold in the case where there is no control in the damping term $m_j \times (m_j^{-1} \times u^j)$, since neglecting this term yields that the first term of the right hand side of equation (1.7) vanishes.

In order to study possible orthogonality, the indicator $\prec m^i_h, u^j_h \succ$ is defined by

$$\prec m^i_h, u^j_h \succ := \frac{1}{\text{amount}} \sum_{l=1}^{L} |\langle m^i_h(x_l), u^j_h(x_l) \rangle|,$$

where $x_l$ is a node, $L$ is the number of nodes, and amount stands for the total amount of summands, where neither $m^i_h(x_l)$ nor $u^j_h(x_l)$ is equal to 0. This indicator can be interpreted as the scalar product of the normalized vectors, which is averaged over space.

As a second indicator $\prec \ldots \succ_r$ we consider the average also in time and define

$$\prec M_h, U_h \succ_r := \frac{1}{T/k} \sum_{j=r+1}^{T/k} \prec m^{j-r}_h, u^j_h \succ,$$

where the index $r \in \{0, 1\}$ determines the shift.

As it is evidenced in Table 1, the optimal state and the control are almost orthogonal in the simulations if there is no control in the damping term (DC = no). This observation is valid for both, the SI scheme and the Projected method. Moreover, this result just slightly differs for different choices $\lambda_1 \in \{0, 10^{-4}\}$.

An interesting observation is the additional approximate orthogonality of the state $M^*_h$ and the adjoint variable $Z_h$ for the Projected method; see Table 1 and Figure 4 (right). The orthogonality property was already motivated by the discussion around (8.4), and is due to the improved approximation of $S^2$ by the (optimal) state through the projection step. Note that this property is not shared from the SI scheme.

<table>
<thead>
<tr>
<th>DC</th>
<th>$\lambda_1$</th>
<th>$\prec M^*_h, Z_h \succ_0$</th>
<th>$\prec M^<em>_h, U^</em>_h \succ_0$</th>
<th>$\prec M^<em>_h, U^</em>_h \succ_1$</th>
<th>$\prec Z_h, U^*_h \succ_1$</th>
</tr>
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<tbody>
<tr>
<td>SI scheme:</td>
<td>yes</td>
<td>0</td>
<td>0.7071</td>
<td>0.1120</td>
<td>0.1119</td>
</tr>
<tr>
<td></td>
<td>yes</td>
<td>$10^{-4}$</td>
<td>0.6772</td>
<td>0.0971</td>
<td>0.0973</td>
</tr>
<tr>
<td></td>
<td>no</td>
<td>0</td>
<td>0.7055</td>
<td>0.0426</td>
<td>0.0430</td>
</tr>
<tr>
<td></td>
<td>no</td>
<td>$10^{-4}$</td>
<td>0.6754</td>
<td>0.0291</td>
<td>0.0295</td>
</tr>
<tr>
<td>Projected:</td>
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<td>0</td>
<td>0.0075</td>
<td>0.1655</td>
<td>0.1664</td>
</tr>
<tr>
<td></td>
<td>yes</td>
<td>$10^{-4}$</td>
<td>0.0061</td>
<td>0.1674</td>
<td>0.1688</td>
</tr>
<tr>
<td></td>
<td>no</td>
<td>0</td>
<td>0.0058</td>
<td>0.0334</td>
<td>0.0341</td>
</tr>
<tr>
<td></td>
<td>no</td>
<td>$10^{-4}$</td>
<td>0.0053</td>
<td>0.0303</td>
<td>0.0311</td>
</tr>
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Table 1. Computed values of $\prec \ldots \succ_r$ for the SI scheme and the Projected method in both cases, with and without control in the damping term (DC).
8.3.3. Rate of convergence of the time-discretization for the SI scheme. In order to study the rate of convergence of the time-discretization, we consider different approximations of the solution \((\M_h^{*,i}, \U_h^{*,i})\) together with \(Z_h^i\), using different time-discretization parameters \(k_i \in \{0.001, 0.002, 0.004, 0.008\}\), while keeping the space-discretization parameter \(h = 0.05\) fixed. The semi-discrete \(L^2(L^2)\)-norm, and the \(L^\infty(L^2)\)-norm of the error between each approximation and an approximation \((\M_h^{*,ex}, \U_h^{*,ex})\) together with \(Z_h^{ex}\) are stated in Table 2, which is simulated with a finer time-discretization parameter \(k_{ex} = 0.0005\). Rates of convergence for the state, the control and the adjoint variable are considered in \(L^2(L^2)\) and \(L^\infty(L^2)\), which are given in Table 2 and Figure 3. We obtain convergence rates in \(L^\infty(L^2)\) for the state, control and adjoint variable close to 0.8.

<table>
<thead>
<tr>
<th>(k)</th>
<th>0.001</th>
<th>0.002</th>
<th>0.004</th>
<th>0.008</th>
<th>(\text{rate})</th>
</tr>
</thead>
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<tr>
<td>(|\M_h^{<em>,i} - \M_h^{</em>,ex}|_{L^\infty(L^2)})</td>
<td>0.0367</td>
<td>0.0814</td>
<td>0.1323</td>
<td>0.1889</td>
<td>0.7874</td>
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<td>(|\U_h^{<em>,i} - \U_h^{</em>,ex}|_{L^\infty(L^2)})</td>
<td>0.3919</td>
<td>0.7559</td>
<td>1.1289</td>
<td>2.0010</td>
<td>0.7840</td>
</tr>
<tr>
<td>(|Z_h^i - Z_h^{ex}|_{L^\infty(L^2)})</td>
<td>0.0225</td>
<td>0.0514</td>
<td>0.0844</td>
<td>0.1131</td>
<td>0.7761</td>
</tr>
<tr>
<td>(|\M_h^{<em>,i} - \M_h^{</em>,ex}|_{L^2(L^2)})</td>
<td>0.0256</td>
<td>0.0577</td>
<td>0.0981</td>
<td>0.1532</td>
<td>0.8597</td>
</tr>
<tr>
<td>(|\U_h^{<em>,i} - \U_h^{</em>,ex}|_{L^2(L^2)})</td>
<td>0.1609</td>
<td>0.3123</td>
<td>0.4530</td>
<td>0.6545</td>
<td>0.6746</td>
</tr>
<tr>
<td>(|Z_h^i - Z_h^{ex}|_{L^2(L^2)})</td>
<td>0.0125</td>
<td>0.0286</td>
<td>0.0482</td>
<td>0.0647</td>
<td>0.7889</td>
</tr>
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</table>

*Table 2.* \(L^2(L^2)\) and \(L^\infty(L^2)\)-errors for the state \(\M_h^*\), control \(\U_h^*\) and adjoint \(Z_h\) variables and corresponding rates of convergence using the SI scheme.
8.4. **Simulations using** $\lambda_0 = 0.02$. We now restrict possible controls by increasing the prefactor $\lambda_0$ to 0.02 in the functional. Figure 6 shows the time-evolution of the components of the functional. If compared with the case of $\lambda_0 = 0.005$ (cf. Figure 2), one observes that the stronger penalization of the $L^2(\Omega)$-norm for the control does not allow to force the state to be close to the desired state any more in the beginning. This can also be observed in Figure 7 where the state and the control variable are shown at the time points $t = 0.2$ and $t = 0.3$. 

![Figure 5](image-url)  
**Figure 5.** $L^\infty(L^2)$-error behaviour of the state, control and adjoint variable for the SI scheme.
Figure 6. Case $\lambda_0 = 0.02$: Functional and components of the functional compared for the SI scheme and the Projected method.

Figure 7. Case $\lambda_0 = 0.02$: Optimal pair $(M^*_h, U^*_h)$ (black, blue) simulated using the SI scheme.

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