Deutsche Forschungsgemeinschaft

Priority Program 1253

Optimization with Partial Differential Equations

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May 2009

Preprint-Number SPP1253-067

Deutsche Forschungsgemeinschaft DFG

http://www.am.uni-erlangen.de/home/spp1253

Optimal control of the Stokes equations: Conforming and non-conforming finite element methods under reduced regularity

Serge Nicaise* Dieter Sirch†

May 28, 2009

Abstract This paper deals with a control-constrained linear-quadratic optimal control problem governed by the Stokes equations. It is concerned with situations where the gradient of the velocity field is not bounded. The control is discretized by piecewise constant functions. The state and the adjoint state are discretized by finite element schemes that are not necessarily conforming. The approximate control is constructed as projection of the discrete adjoint velocity in the set of admissible controls. It is proved that under certain assumptions on the discretization of state and adjoint state this approximation is of order 2 in $L^2(\Omega)$. As first example a prismatic domain with a reentrant edge is considered where the impact of the edge singularity is counteracted by anisotropic mesh grading and where the state and the adjoint state are approximated in the lower order Crouzeix-Raviart finite element space. The second example concerns a nonconvex, plane domain, where the corner singularity is treated by isotropic mesh grading and state and adjoint state can be approximated by a couple of standard element pairs.

Key Words PDE-constrained optimization, control-constraints, finite element method, non-conforming elements, anisotropic mesh-grading, a priori error estimates, Stokes equations

AMS subject classification 65N30, 49M25

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1 Introduction

In this paper we consider the optimal control problem

$$\min J(v,u) = \frac{1}{2}\|v - v_d\|^2_{L^2(\Omega)^d} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \tag{1.1}$$

subject to

$$\begin{aligned} -\Delta v + \nabla p &= u && \text{in } \Omega, \\ \nabla \cdot v &= 0 && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

and subject to the pointwise control constraints

$$u_a \le u(x) \le u_b \text{ for a.a. } x \in \Omega. \tag{1.3}$$

Here, Ω is an open and bounded domain in $\mathbb{R}^d$ with $d = 2$ or $d = 3$. $\partial\Omega$ is the boundary of Ω. Further, the quantities $u_a, u_b \in \mathbb{R}^d$ are constant vectors and the regularization parameter ν is a fixed positive number. The desired velocity field v_d is assumed to be from $C^{0,\sigma}(\bar{\Omega})^d$, $\sigma \in (0,1)$. We introduce the space $U := L^2(\Omega)^d$ and the set of admissible controls

$$U^{\mathrm{ad}} = \{u \in U : u_a \le u \le u_b \text{ a.e.}\}.$$

This paper investigates the discretization of the optimal control problem (1.1)–(1.3) based on a finite element approximation of the state and the control variable. The discussion of discretizations of optimal control problems governed by partial differential equations started with papers of Falk [18], Gevici [19] and Malanowski [26]. In the past few years several results concerning a priori error estimates for this type of problem were proved, see e.g. [9, 14, 13, 12, 34, 30, 33]. These papers are either concerned with linear and quasi-linear elliptic or linear parabolic state equations. The authors established convergence rates for the error in the control of 1 and $\frac{3}{2}$ in $L^2(\Omega)$ and of 1 in $L^\infty(\Omega)$. These results could be improved by two new discretization concepts, namely the variational discrete approach and the post-processing approach. In both cases, linear-quadratic optimal control problems governed by an elliptic equation were considered first. In the variational discrete approach of Hinze [23] the control is not discretized but approximated by the use of the first order optimality condition and the discretized state and adjoint state. For this approximation convergence order 2 in $L^2(\Omega)$ was proved under the assumption of H^2-regularity. Meyer and Rösch [29] used a post-processing technique to get an improved approximation for the control. The optimal control is discretized by piecewise constant functions, state and adjoint state by piecewise linear functions. The so computed adjoint state is projected in the set of admissible controls and yields an approximation of the optimal control that is not in the finite element space anymore. They showed that for this approximation the L^2-error behaves like h^2 provided the state variable is contained in $H^2(\Omega)$. Apel and co-authors got the same result for situations with reduced regularity in the state caused by corners and/or edges in the domain Ω.

They counteracted the singularities by isotropic [5, 8] or anisotropic mesh grading [7]. In [4] the authors proved a convergence rate of $h^2 \ln h$ in $L^\infty(\Omega)$ in plane domains for both, the post-processed as well as the variational-discrete control.

Rösch and Vexler applied the post-processing technique to a linear-quadratic optimal control problem governed by the Stokes equations [35]. They proved second order convergence under the assumption, that the velocity field admits full regularity, what means it is contained in $H^2(\Omega) \cap W^{1,\infty}(\Omega)$. We should also mention that several articles were published for the optimal control of the Stokes and Navier-Stokes equation without control constraints, see e.g. [21, 22, 10, 17].

This paper extends the results of Rösch and Vexler [35] and Apel et al. [5, 8, 7] to the optimal control of the Stokes equations under weaker regularity assumptions. This means, we do not assume that the velocity field is contained in $W^{1,\infty}(\Omega)^d \cap H^2(\Omega)^d$, but only in some weighted space $H^2_\omega(\Omega)$ (comp. (2.7)). In [35] the authors made use of the $W^{1,\infty}$-regularity of the velocity field in an explicit manner. Therefore they restrict theirselves to polygonal, convex domains $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and assume in the case $d = 3$ that the edge openings of the domain Ω are smaller than $\frac{2}{3}\pi$. The authors of [5, 8, 7] considered optimal control problems with scalar elliptic state equations also in non-convex domains, such that the state is not contained in $W^{1,\infty}(\Omega) \cap H^2(\Omega)$. We do not only combine the techniques developed in these papers but also introduce substantially new things. First of all, we allow the discretization of the velocity field to be non-conforming. To the authors' best knowledge this is the first time that non-conforming finite element methods are investigated in the context of optimal control. Furthermore we prove second order convergence for the finite element approximation of the velocity field of the state equation in $L^2(\Omega)$ under general assumptions on the finite dimensional spaces X_h and M_h. Such assumptions are e.g. discrete Poincaré inequality, consistency and uniform inf-sup-condition. They are valid for many element pairs on isotropic as well as anisotropic meshes (see e.g. [2], [11]). Under these assumptions we prove the supercloseness result,

$$\|\bar{u}_h - R_h \bar{u}\|_{L^2(\Omega)} \leq ch^2, \tag{1.4}$$

where $\bar{u}_h$ is the solution of the discretized optimality system and R_h an operator that maps continuous functions to the space of piecewise constant functions. As in previous publications concerning the post-processing approach for control-constrained optimal control problems this is the key to the proof of the main result, namely the superconvergence of the approximate control $\tilde{u}_h$ to the optimal control $\bar{u}$,

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Omega)} \leq ch^2. \tag{1.5}$$

Here, $\tilde{u}_h$ is the projection of the approximate adjoint velocity in the set of admissible controls U^{ad}. In the last two sections we check the general assumptions for concrete examples. We first consider a prismatic domain $\Omega = G \times Z$, where $G \subset \mathbb{R}^2$ is a two-dimensional domain with a reentrant corner and $Z \subset \mathbb{R}$ an interval. The domain is discretized by anisotropic tetrahedral elements, whose size depends on the distance to the reentrant edge. The velocity is approximated by Crouzeix-Raviart elements, for the pressure we use piecewise constant functions. The control is also approximated by piecewise constant functions. Notice, that the proofs in [7] cannot be adapted in a straightforward manner. The reason is not only the more complicated structure of the Stokes equations but also the fact, that the authors of [7] used the additional regularity of the solution and its derivatives in edge-direction in $L^p(\Omega)$ for general p. In the case of the Stokes equations such results are only available for $p = 2$, see [3]. This requires a modification in the proof of the assumption (A9) below. Our second example concerns a two-dimensional setting, where the domain has a reentrant corner. We prove that our general assumptions are satisfied for a couple of element pairs as long as one uses a mesh that is tailored to the corner singularity.

Let us give a short outline of the paper. In Section 2 we extract a couple of general assumptions on the discretization of the optimal control problem, that allows us to prove estimates (1.4) and (1.5). In Section 3 we give some estimates concerning the finite element error in the state equation. The main results (1.4) and (1.5) are stated more precisely and proved in Section 4. Furthermore we prove second order convergence in the optimal velocity and optimal adjoint velocity as well as first order convergence in the optimal pressure and optimal adjoint pressure. As already mentioned above Section 5 and Section 6 contain two particular configurations, namely an anisotropic and non-conforming discretization in a prismatic domain and a isotropic discretization for a couple of element pairs in a polygonal domain. For both settings we illustrate our theoretical findings by numerical examples.

2 General assumptions on the discretization

In the following we consider solutions of the Stokes equations (1.2) in the sense of a weak formulation. Therefore we introduce the spaces

$$X = \left\{ v \in (H^1(\Omega))^d : v|_{\partial\Omega} = 0 \right\},$$
$$M = \left\{ p \in L^2(\Omega) : \int_\Omega p = 0 \right\}$$

and the bilinear forms $a : X \times X \to \mathbb{R}$ and $b : X \times M \to \mathbb{R}$ as

$$a(v, \varphi) := \sum_{i=1}^{d} \int_\Omega \nabla v_i \cdot \nabla \varphi_i \quad \text{and} \quad b(\varphi, p) := -\int_\Omega p \nabla \cdot \varphi,$$

respectively. We write $(\cdot, \cdot)$ for the inner product in $L^2(\Omega)$ or $L^2(\Omega)^d$ according to the context. Then the weak solution $(v, p) \in X \times M$ of equation (1.2) is given as unique solution of

$$a(v, \varphi) + b(\varphi, p) = (u, \varphi) \qquad \forall \varphi \in X \tag{2.1}$$
$$b(v, \psi) = 0 \qquad \forall \psi \in M. \tag{2.2}$$

For the formulation of the optimality system we introduce the adjoint equation

$$\begin{aligned} -\Delta w - \nabla r &= v - v_d && \text{in } \Omega, \\ \nabla \cdot w &= 0 && \text{in } \Omega, \\ w &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Its weak formulation is given as

$$\begin{aligned} a(w,\varphi) - b(\varphi, r) &= (v - v_d, \varphi) && \forall \varphi \in X \\ b(w,\psi) &= 0 && \forall \psi \in M. \end{aligned}$$

We introduce the solution mappings S and S^p of the continuous state equation such that there holds for all $(\varphi,\psi) \in X \times M$ and $u \in U$

$$a(Su,\varphi) + b(\varphi, S^p u) = (u,\varphi) \quad \text{and} \quad b(Su,\psi) = 0.$$

Analogously, we introduce S^* and $S^{p,*}$ as solution mappings of the adjoint equation. Further we define the operator P such that $Pu = (S^*(Su - v_d)) = w$. Moreover, we introduce the projection

$$\Pi_{[u_a,u_b]} f(x) := \max(u_a, \min(u_b, f(x)).$$

The optimal control problem (1.1) – (1.3) is strictly convex and radially unbounded. Therefore it admits a unique solution. The first order necessary condition can be formulated as variational inequality and is also sufficient for optimality. These statements are summarized in the following lemma. A proof can be found e.g. in [26].

Lemma 2.1. *The optimal control problem* (1.1) – (1.3) *has a unique solution* $\bar{u}$. *The variational inequality*

$$(\bar{w} + \nu \bar{u}, u - \bar{u})_U \geq 0 \qquad \forall u \in U^{\text{ad}} \tag{2.3}$$

is a necessary and sufficient condition for the optimality of $\bar{u}$ *with associated state* $\bar{y} = (\bar{v}, \bar{p})$ *and associated adjoint state* $\bar{z} = (\bar{w}, \bar{r})$. *The projection formula*

$$\bar{u} = \Pi_{[u_a,u_b]} \left(-\frac{1}{\nu} \bar{w} \right) \tag{2.4}$$

is an equivalent formulation for condition (2.3).

In the remainder of this section we give a couple of assumptions which are sufficient for proving a finite element error estimate for the optimal control problem (1.1) – (1.3). In order to discretize the optimal control problem, we consider a conforming triangulation $\mathcal{T}_h$ of Ω in the sense of Ciarlet [15], i.e.

(A1)

- $\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} \bar{T}$

- For two arbitrary elements $T_1, T_2 \in \mathcal{T}_h$ with $T_1 \neq T_2$ one has $T_1 \cap T_2 = \emptyset$.
- Any face of any element $T_1 \in \mathcal{T}_h$ is either a subset of the boundary $\partial\Omega$ or a face of another cell $T_2 \in \mathcal{T}_h$.

The control variable u is approximated by piecewise constant functions using the discrete spaces

$$U_h = \Big\{ u_h \in U : u_h|_T \in (\mathcal{P}_0)^d \text{ for all } T \in \mathcal{T}_h \Big\} \quad \text{and} \quad U_h^{\mathrm{ad}} = U_h \cap U^{\mathrm{ad}}$$

For the discretization of the state equation we assume a given velocity approximation space X_h and a given pressure approximation space M_h each consisting of piecewise polynomial functions, such that $M_h \subset M$ but not necessarily $X_h \subset X$. Since the velocity space X may not be included in the discrete velocity space X_h, we define the approximate solution of the state equation (2.1)–(2.2) by using the weaker bilinear forms $a_h : X_h \times X_h \to \mathbb{R}$ and $b_h : X_h \times M_h \to \mathbb{R}$ with

$$a_h(v_h, \varphi_h) := \sum_{T \in \mathcal{T}_h} \sum_{i=1}^{d} \int_T \nabla v_{h,i} \cdot \nabla \varphi_{h,i} \quad \text{and} \quad b_h(\varphi_h, p_h) := -\sum_{T \in \mathcal{T}_h} \int_T p_h \nabla \cdot \varphi_h.$$

Here, the i-th component of the vectors v_h and φ_h is denoted by $v_{h,i}$ and $\varphi_{h,i}$, respectively. The bilinear form $a_h(\cdot,\cdot)$ induces a broken $H^1(\Omega)$-norm by $\|\cdot\|_{X_h} := a_h(\cdot,\cdot)^{1/2}$. For a given control $u \in U$ the discretized state equation reads as

Find $(v_h, p_h) \in X_h \times M_h$ such that

$$a_h(v_h, \varphi_h) + b_h(\varphi_h, p_h) = (u, \varphi_h) \qquad \forall \varphi_h \in X_h \tag{2.5}$$

$$b_h(v_h, \psi_h) = 0 \qquad \forall \psi_h \in M_h. \tag{2.6}$$

Analogously, the discretized adjoint equation is given as

Find $(w_h, r_h) \in X_h \times M_h$ such that

$$\begin{aligned} a_h(w_h, \varphi_h) - b_h(\varphi_h, r_h) &= (v_h - v_d, \varphi_h) \qquad && \forall \varphi_h \in X_h \\ b_h(w_h, \psi_h) &= 0 && \forall \psi_h \in M_h. \end{aligned}$$

The solution mappings S_h and S_h^p of the discretized state equation are defined such that one has for all $(\varphi_h, \psi_h) \in X_h \times M_h$ and $u \in U$

$$a_h(S_h u, \varphi_h) + b_h(\varphi_h, S_h^p u) = (u, \varphi_h) \quad \text{and} \quad b(S_h u, \psi_h) = 0.$$

Analogously, we introduce S_h^* and $S_h^{p,*}$ as solution mappings of the discrete adjoint equation and the operator P_h such that $P_h u = S_h^*(S_h u - v_d) = w_h$.

The discretized optimal control problem reads as

$$\begin{aligned} J_h(\bar{u}_h) &= \min_{u_h \in U_h^{\mathrm{ad}}} J_h(u_h) \\ J_h(u_h) &:= \frac{1}{2}\|S_h u_h - v_d\|_{L^2(\Omega)} + \frac{\nu}{2}\|u_h\|_{L^2(\Omega)}. \end{aligned}$$

As in the continuous case, this is a strictly convex and radially unbounded optimal control problem, that admits a unique solution $\bar{u}_h$. This solution satisfies the necessary and sufficient optimality conditions

$$\begin{aligned} \bar{v}_h &= S_h \bar{u}_h, \\ \bar{w}_h &= S_h^*(\bar{v}_h - v_d), \\ (\nu \bar{u}_h + \bar{w}_h, u_h - \bar{u}_h)_U &\geq 0 \quad \forall u_h \in U_h^{\text{ad}}. \end{aligned}$$

In the following we state assumptions that has to be satisfied by the spaces M_h and X_h. In the Sections 5 and 6 we present examples of suitable spaces.

Since we also like to include non-convex domains Ω, it is convenient to describe the regularity of the solution in weighted Sobolev spaces. Since we may consider problems with corner- and/or edge-singularities, we introduce the general weighted Sobolev spaces $H^k_\omega(\Omega)^d$, $k = 1, 2$. The corresponding norm is defined as

$$\|v\|_{H^k_\omega(\Omega)^d} = \left(\sum_{|\alpha| \leq k} \|\omega_\alpha D^\alpha v\|^2_{L^2(T)^d} \right)^{1/2} \tag{2.7}$$

where ω_α is a suitable positive weight depending on the concrete problem under consideration.

(A2) For the solution of the Stokes problem one has for a sufficiently smooth right-hand side u

$$(v, p) \in H^2_{\omega_1}(\Omega)^d \times H^1_{\omega_2}(\Omega)$$

with suitable weights ω_1 and ω_2.

(A3) The discrete Poincaré inequality

$$\|v_h\|_{L^2(\Omega)^d} \leq c\|v_h\|_{X_h} \qquad \forall v_h \in X_h$$

holds.

(A4) The embedding $H^2_\omega(\Omega) \hookrightarrow C(\bar{\Omega})$ holds such that $S : U \to C(\bar{\Omega})^d$.

(A5) There exist interpolation operators $i_h^v : H^2_\omega(\Omega)^d \cap X \to X_h \cap X$ and $i_h^p : H^1_\omega(\Omega) \cap M \to M_h$ such that for the solution $(v, p) \in X \times M$ of the Stokes problem (1.2) the interpolation properties

(i) $\|v - i_h^v v\|_{X_h} \leq ch\|v\|_{H^2_\omega(\Omega)} \leq ch\|u\|_{L^2(\Omega)^d}$

(ii) $\|v - i_h^v v\|_{L^\infty(\Omega)} \leq c\|u\|_{L^2(\Omega)^d}$

(iii) $\|p - i_h^p p\|_{L^2(\Omega)} \leq ch\|p\|_{H^1_\omega(\Omega)} \leq ch\|u\|_{L^2(\Omega)^d}$

hold.

(A6) There exists a p satisfying

$$\begin{aligned} p &< \infty \quad \text{if } d = 2 \\ p &\leq 6 \quad \text{if } d = 3, \end{aligned} \tag{2.8}$$

such that the inverse estimate

$$\|\varphi_h\|_{L^\infty(\Omega)} \leq ch^{-1}\|\varphi_h\|_{L^p(\Omega)} \qquad \forall \varphi_h \in X_h$$

is valid.

(A7) A consistency error estimate holds for the space X_h,

$$|a_h(v, \varphi_h) + b_h(\varphi_h, p) - (u, \varphi_h)| \leq ch\|\varphi_h\|_{X_h}\|u\|_{L^2(\Omega)} \qquad \forall (u, \varphi_h) \in L^2(\Omega) \times X_h.$$

where $(v, p) \in X \times M$ is the solution of the Stokes problem (1.2).

(A8) The pair (X_h, M_h) fulfills the uniform discrete inf-sup-condition, i.e. there exists a positive constant β independent of h such that

$$\inf_{\psi_h \in M_h} \sup_{\varphi_h \in X_h} \frac{b(\varphi_h, \psi_h)}{\|\varphi_h\|_{X_h}\|\psi\|_M} \geq \beta.$$

We introduce two projection operators. For continuous functions f we define the projection in the space U_h of piecewise constant functions by

$$(R_h f)(x) := f(S_T) \text{ if } x \in T \tag{2.9}$$

where S_T denotes the centroid of the element T. The operator Q_h projects L^2-functions g in the space U_h of piecewise constant functions,

$$(Q_h g)(x) := \frac{1}{|T|}\int_T g(x)\,\mathrm{d}x \text{ for } x \in T. \tag{2.10}$$

Both operators are defined componentwise for vector valued functions. For these operators we state the following two assumptions.

(A9) The optimal control $\bar{u}$ and the corresponding adjoint velocity $\bar{w}$ satisfy the inequality

$$\|Q_h\bar{w} - R_h\bar{w}\|_{L^2(\Omega)} \leq ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})}\right).$$

(A10) For the optimal control $\bar{u}$ and all functions $\varphi_h \in X_h$ the inequality

$$(Q_h\bar{u} - R_h\bar{u}, \varphi_h)_{L^2(\Omega)} \leq ch^2\|\varphi_h\|_{L^\infty(\Omega)}\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)$$

holds.

The assumptions (A9) and (A10) are independent of the spaces M_h and X_h. They only depend on the structure of the underlying mesh as well as on the regularity of the solution of the optimal control problem.

3 Results from finite element theory

In this section we give some estimates concerning the finite element error in the state equation. We especially pay attention to non-conforming methods and prove that the $L^2(\Omega)$-error of the approximation of the velocity field is of order 2 under the rather general assumptions (A1)–(A8).

Lemma 3.1. *Assume that assumptions (A1)–(A8) hold. For an arbitrary control $u \in U$ the approximation error in the state and adjoint state can be estimated by*

$$\|S_h^p u - S^p u\|_{L^2(\Omega)} + \|S_h u - Su\|_{X_h} \le ch\|u\|_U \tag{3.1}$$

$$\|S_h u - Su\|_U \le ch^2\|u\|_U \tag{3.2}$$

$$\|S_h u - Su\|_{L^p(\Omega)} \le h\|u\|_U \text{ for } p \text{ satisfying (2.8)} \tag{3.3}$$

$$\|S_h u - Su\|_{L^\infty(\Omega)} \le c\|u\|_U \tag{3.4}$$

$$\|P_h u - Pu\|_U \le ch^2(\|u\|_U + \|v_d\|_U). \tag{3.5}$$

Proof. In this proof we use the abbreviation $v = Su$, $v_h = S_h u$, $w = Pu$ and $w_h = P_h u$. From [11, Chap. II, Proposition 2.16] one has

$$\|v - v_h\|_{X_h} + \|p - p_h\|_{L^2(\Omega)} \le c \inf_{\varphi_h \in X_h} \|v - \varphi_h\|_{X_h} + c \inf_{q_h \in M_h} \|p - q_h\|_{L^2(\Omega)} + c \sup_{\varphi_h \in X_h} \frac{|a_h(v, \varphi_h) + b_h(\varphi_h, p) - (f, v_h)|}{\|\varphi_h\|_{X_h}}$$

Then estimate (3.1) can be concluded from (A5) and (A7).

In order to prove the L^2-error estimate (3.2) we apply a non-conforming version of the Aubin-Nitsche method. Therefore we consider for $g \in U$ the solution $(\varphi_g, \psi_g) \in (H_0^1(\Omega)^d \cap H_\omega^2(\Omega)^d) \times (L^2(\Omega) \cap H_\omega^1(\Omega))$ of the saddle-point problem

$$a(\varphi_g, \varphi) + b(\varphi, \psi_g) = (g, \varphi) \qquad \forall \varphi \in X \tag{3.6}$$

$$b(\varphi_g, \psi) = 0 \qquad \forall \psi \in M. \tag{3.7}$$

We introduce for $(\varphi, p, v) \in X \times M \times X$ the abbreviation

$$d_h(\varphi, p, v) := a_h(\varphi, v) + b_h(v, p) - (u, v).$$

Then one has for $\varphi_h \in X_h$ and $\psi_h \in M_h$

$$\begin{aligned}
& a_h(v-v_h,\varphi_g-\varphi_h)+b_h(v-v_h,\psi_g-\psi_h)+b_h(\varphi_g-\varphi_h,p-p_h) \\
& \quad -d_h(v,p,\varphi_g-\varphi_h)-d_h(\varphi_g,\psi_g,v-v_h) \\
&= -a_h(v-v_h,\varphi_h)-b_h(v-v_h,\psi_h)-b_h(\varphi_g-\varphi_h,p_h)- \\
& \quad a_h(v,\varphi_g)+a_h(v,\varphi_h)+(u,\varphi_g-\varphi_h)+(g,v-v_h) \\
&= \ a_h(v_h,\varphi_h)-(u,\varphi_h)-a_h(v,\varphi_g)+(u,\varphi_g)- \\
& \quad b_h(\varphi_g-\varphi_h,p_h)-b_h(v-v_h,\psi_h)+(g,v-v_h) \\
&= -b_h(\varphi_h,p_h)+b(\varphi_g,p)-b_h(\varphi_g-\varphi_h,p_h)-b_h(v-v_h,\psi_h)+(g,v-v_h) \\
&= -b_h(\varphi_g,p_h)+b_h(v_h,\psi_h)-b_h(v,\psi_h)+(g,v-v_h) \\
&= \quad (g,v-v_h)
\end{aligned}$$

In the last two steps we have used (3.7) and (2.6), respectively. Further, $b_h(v,\psi_h)=0$ since $M_h \subset M$. Now we can continue with

$$\begin{aligned}
\|v-v_h\|_{L^2(\Omega)^d} &= \sup_{0\neq g\in L^2(\Omega)^d} \frac{(g,v-v_h)}{\|g\|_{L^2(\Omega)^d}} \\
&\le \sup_{0\neq g\in L^2(\Omega)^d} \|g\|^{-1}_{L^2(\Omega)^d}\left(|a_h(v-v_h,\varphi_g-\varphi_h)|+|b_h(v-v_h,\psi_g-\psi_h)|+\right. \\
&\quad \left. +\,|b_h(\varphi_g-\varphi_h,p-p_h)|+|d_h(v,p,\varphi_g-\varphi_h)|+|d_h(\varphi_g,\psi_g,v-v_h))\right|.
\end{aligned} \tag{3.8}$$

We estimate these terms separately. For the first term we set $\varphi_h = i_h^v\varphi_g$, where i_h^v is the interpolation operator of (A5). This yields

$$\begin{aligned}
|a_h(v-v_h,\varphi_g-i_h^v\varphi_g)| &\le c\|v-v_h\|_{X_h}\|\varphi_g-i_h^v\varphi_g\|_{X_h} \\
&\le ch^2\|u\|_{L^2(\Omega)^d}\|g\|_{L^2(\Omega)^d}.
\end{aligned} \tag{3.9}$$

To estimate the second term we set $\psi_h = i_h^p\psi_g$ with the operator i_h^p of (A5). Then one has

$$\begin{aligned}
|b_h(v-v_h,\psi_g-i_h^p\psi_g)| &\le \|v-v_h\|_{X_h}\|\psi_g-i_h^p\psi_g\|_{L^2(\Omega)} \\
&\le ch^2\|u\|_{L^2(\Omega)^d}\|g\|_{L^2(\Omega)^d}.
\end{aligned} \tag{3.10}$$

where we have used (3.1) and (A5)(ii). The third term can be estimated by

$$\begin{aligned}
|b_h(\varphi_g-i_h^v\varphi_g,p-p_h)| &\le \|\varphi_g-i_h^v\varphi_g\|_{X_h}\|p-p_h\|_{L^2(\Omega)} \\
&\le ch^2\|u\|_{L^2(\Omega)^d}\|g\|_{L^2(\Omega)^d}
\end{aligned} \tag{3.11}$$

where we used the properties of i_h^v given in (A5) and the L^2-error estimate for p in (3.1). Since $\varphi_g - i_h^v\varphi_g \in X$ there holds for the fourth term

$$d_h(v,p,\varphi_g-i_h^v\varphi_g) = a_h(v,\varphi_g-i_h^v\varphi_g)+b_h(\varphi_g-i_h^v\varphi_g,p)-(u,\varphi_g-i_h^v\varphi_g)=0. \tag{3.12}$$

Finally, the fifth term yields

$$\begin{aligned}|d_h(\varphi_g, \psi_g, v - v_h)| &= |a_h(\varphi_g, v - v_h) + b_h(v - v_h, \psi_g) - (g, v - v_h)| \\ &\le |a_h(\varphi_g, v - i_h^v v) + b_h(v - i_h^v v, \psi_g) - (g, v - i_h^v v)| \\ &\quad + |a_h(\varphi_g, i_h^v v - v_h) + b_h(i_h^v v - v_h, \psi_g) + (g, i_h^v v - v_h)|. \end{aligned} \tag{3.13}$$

Since $v - i_h^v v \in X$ we can conclude like above

$$|a_h(\varphi_g, v - i_h^v v) + b_h(v - i_h^v v, \psi_g) - (g, v - i_h^v v)| = 0. \tag{3.14}$$

The consistency error estimate of (A7) entails

$$|a_h(\varphi_g, i_h^v v - v_h) + b_h(i_h^v v - v_h, \psi_g) + (g, i_h^v v - v_h)| \le ch\|i_h^v v - v_h\|_{X_h}\|g\|_{L^2(\Omega)}. \tag{3.15}$$

With equations (3.14) and (3.15) we can continue from (3.13) with

$$\begin{aligned}|d_h(\varphi_g, \psi_g, v - v_h)| &\le ch\|i_h^v v - v_h\|_{X_h}\|g\|_{L^2(\Omega)} \\ &\le ch(\|v - v_h\|_{X_h} + \|v - i_h^v v\|_{X_h})\|g\|_{L^2(\Omega)} \\ &\le ch^2\|u\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)}\end{aligned}$$

where we have used again (3.1) and (A5)(i). This last estimate implies together with (3.8)–(3.12) the assertion (3.2).

Estimate (3.3) follows directly from inequality (3.1) by the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for p satisfying (2.8).

In order to prove inequality (3.4), we can conclude from the triangle inequality and properties (A5) and (A6)

$$\begin{aligned}\|(S_h - S)u\|_{L^\infty(\Omega)^d} &\le \|Su - i_h^v Su\|_{L^\infty(\Omega)^d} + \|S_h u - i_h^v Su\|_{L^\infty(\Omega)^d} \\ &\le c\|u\|_{L^2(\Omega)^d} + ch^{-1}\|S_h u - i_h^v Su\|_{L^p(\Omega)^d} \\ &\le c\|u\|_{L^2(\Omega)^d} + ch^{-1}\left(\|S_h u - Su\|_{L^p(\Omega)^d} + \|Su - i_h^v Su\|_{L^p(\Omega)^d}\right)\end{aligned} \tag{3.16}$$

for a certain p satisfying (2.8). Since $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for such a p one can conclude from (A5)(i) that

$$\|Su - i_h^v Su\|_{L^p(\Omega)^d} \le ch\|u\|_U.$$

With this estimate we can continue from (3.16) and get together with inequality (3.3) the desired result (3.4).

For the proof of inequality (3.5), we write

$$\begin{aligned}\|P_h u - Pu\|_U &= \|S_h^*(S_h u - v_d) - S^*(Su - v_d)\|_U \\ &\le \|S^* Su - S_h^* S_h u\|_U + \|S^* v_d - S_h^* v_d\|_U \\ &\le \|S^*\|_{U\to U}\|Su - S_h u\|_U + \|(S^* - S_h^*)S_h u\|_U + \|(S^* - S_h^*)v_d\|_U \\ &\le ch^2\|u\|_U + ch^2\|S_h u\|_U + ch^2\|v_d\|_U\end{aligned} \tag{3.17}$$

where we have used the boundedness of S^*, i.e. $\|S^*\|_{U\to U} \leq c$ and inequality (3.2). Notice, that the proof of (3.1) and (3.2) also works for S^* and S_h^*, respectively. As a direct consequence of (3.1) one has the boundedness $\|S_h^*\|_{U\to U} \leq c$, such that inequality (3.17) yields the assertion (3.5). □

Lemma 3.2. *The discrete solution operators S_h and S_h^* are bounded,*

$$\begin{aligned} \|S_h\|_{U\to U} &\leq c, & \|S_h^*\|_{U\to U} &\leq c, \\ \|S_h\|_{U\to X_h} &\leq c, & \|S_h^*\|_{U\to X_h} &\leq c, \\ \|S_h\|_{U\to L^\infty(\Omega)^d} &\leq c, & \|S_h^*\|_{U\to L^\infty(\Omega)^d} &\leq c, \end{aligned}$$

with constants c independent of h.

Proof. We show this lemma for the operator S_h, the proofs for S_h^* are analogous. The first estimate follows with

$$\|S_h u\|_U \leq \|Su\|_U + \|S_h u - Su\|_U$$

from the boundedness of S as operator from U to U and inequality (3.2). The subtraction of the equations (2.5) and (2.6) yields

$$a_h(S_h u, \varphi_h) + b_h(\varphi_h, p_h) - b_h(S_h u, \psi_h) = (u, \varphi_h) \quad \forall (\varphi_h, \psi_h) \in X_h \times M_h.$$

If one chooses $(\varphi_h, \psi_h) = (S_h u, p_h)$ this implies

$$a_h(S_h u, S_h u) = (u, S_h u).$$

Therefore we can estimate

$$\begin{aligned} \|S_h u\|_{X_h}^2 &= a_h(S_h u, S_h u) = (u, S_h u) & (3.18)\\ &\leq c\|u\|_{L^2(\Omega)}\|S_h u\|_{L^2(\Omega)} \leq c\|u\|_{L^2(\Omega)}\|S_h u\|_{X_h}, & (3.19) \end{aligned}$$

where we have used the Cauchy-Schwarz inequality and the discrete Poincaré inequality (A3). Division by $\|S_h u\|_{X_h}$ yields $\|S_h u\|_{X_h} \leq c\|u\|_{L^2(\Omega)}$ and the second estimate is proved. The third estimate follows from the boundedness of S and inequality (3.4),

$$\|S_h u\|_{L^\infty(\Omega)^d} \leq \|Su - S_h u\|_{L^\infty(\Omega)^d} + \|Su\|_{L^\infty(\Omega)^d} \leq c\|u\|_U.$$

□

4 Superconvergence

In this section the main result of the paper is given. First we prove in Theorem 4.3, that the approximate solution $\bar{u}_h$ is closer (in the L^2-sense) to the interpolant $R_h\bar{u}$ than to the optimal control $\bar{u}$. This was originally discovered by Meyer and Rösch [29] for an optimal control problem governed by the Poisson equation. Such a fact is often referred as *supercloseness.* Based on this result, we show that the approximate control $\tilde{u}_h$, which is constructed as projection of the discrete adjoint velocity $\bar{w}_h$ in the admissible set U^{ad}, converges in $L^2(\Omega)$ to the optimal control $\bar{u}$ with order 2. Due to the fact that $\bar{u}$ was originally approximated by piecewise constant functions, this is a *superconvergence* result.

We first prove some properties of the operator R_h.

Lemma 4.1. *The estimates*

$$\|S_h\bar{u} - S_hR_h\bar{u}\|_U \le ch^2 \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right) \tag{4.1}$$

$$\|P_h\bar{u} - P_hR_h\bar{u}\|_U \le ch^2 \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right) \tag{4.2}$$

are valid.

Proof. A proof of this lemma in case of a conforming discretization was given in [35]. In the context of optimal control of the Poisson equation Apel and Winkler gave a proof in [8]. The following proof is similar to that, but takes a possible non-conformity into account. First of all, we write

$$\begin{aligned}\|S_h\bar{u} - S_hR_h\bar{u}\|_U^2 &= (S_h\bar{u} - S_hR_h\bar{u}, S_h\bar{u} - S_hR_h\bar{u})_U \\ &= (S_h(\bar{u} - R_h\bar{u}), (S_h\bar{u} - v_d) - (S_hR_h\bar{u} - v_d))_U \\ &= (\bar{u} - R_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_U \\ &= (\bar{u} - Q_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_U + (Q_h\bar{u} - R_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_U. \end{aligned} \tag{4.3}$$

We estimate these two terms separately. For the first term, we recall the inequality

$$(f - Q_hf, g)_{L^2(T)} \le ch_T^2|f|_{H^1(T)}|g|_{H^1(T)}$$

which is valid for all $f, g \in H^1(T)$, see [8, (47)]. Therefore one can conclude

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} (\bar{u} - Q_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_{L^2(T)^d} &\le c \sum_{T\in\mathcal{T}_h} h_T^2|\bar{u}|_{H^1(T)^d}|P_h\bar{u} - P_hR_h\bar{u}|_{H^1(T)^d} \\ &\le ch^2|\bar{u}|_{H^1(\Omega)^d} \left(\sum_{T\in\mathcal{T}_h} |P_h\bar{u} - P_hR_h\bar{u}|^2_{H^1(T)^d}\right)^{1/2}. \end{aligned} \tag{4.4}$$

Since one can write

$$\sum_{T\in\mathcal{T}_h} |P_h\bar{u} - P_hR_h\bar{u}|^2_{H^1(T)^d} = \|S_h^*(S_h\bar{u} - S_hR_h\bar{u})\|^2_{X_h} \le \|S_h^*\|^2_{U\to X_h}\|S_h\bar{u} - S_hR_h\bar{u}\|^2_U$$

it follows from Lemma 3.2 and (4.4)

$$(\bar{u} - Q_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_U \le ch^2|\bar{u}|_{H^1(\Omega)^d}\|S_h\bar{u} - S_hR_h\bar{u}\|_U. \tag{4.5}$$

According to the projection formula (2.4) the domain Ω splits in two parts, the inactive part $\mathcal{I}$, where $\bar{u} = -\frac{1}{\nu}\bar{w}$ and the active part $\Omega\backslash\mathcal{I}$, where $\bar{u}$ is constant. Since $|\bar{u}|_{H^1(\Omega\backslash\mathcal{I})^d} = 0$ one has

$$|\bar{u}|_{H^1(\Omega)^d} \leq c\|\bar{w}\|_{H^1(\Omega)^d} \leq c\|S^*\|_{U\to H^1(\Omega)^d}\|S\bar{u} - v_d\|_U \leq c\left(\|\bar{u}\|_{L^\infty(\Omega)^d} + \|v_d\|_{L^\infty(\Omega)^d}\right) \tag{4.6}$$

where we have used the boundedness $\|S^*\|_{U\to H^1(\Omega)^d} \leq c$ and $\|S\|_{U\to U} \leq c$ as well as the embedding $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$. So we get from (4.5) and (4.6) the estimate

$$(\bar{u} - Q_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_U \leq ch^2\left(\|\bar{u}\|_{L^\infty(\Omega)^d} + \|v_d\|_{L^\infty(\Omega)^d}\right)\|S_h\bar{u} - S_hR_h\bar{u}\|_U. \tag{4.7}$$

In order to estimate the second term of equation (4.3), we utilize Assumption (A9) and get

$$\begin{aligned}(Q_h\bar{u} - R_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_U &\leq ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right)\|P_h\bar{u} - P_hR_h\bar{u}\|_{L^\infty(\Omega)^d}\\ &\leq ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right)\|S_h^*\|_{U\to L^\infty(\Omega)^d}\|S_h\bar{u} - S_hR_h\bar{u}\|_U\\ &\leq ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right)\|S_h\bar{u} - S_hR_h\bar{u}\|_U \end{aligned} \tag{4.8}$$

by applying Lemma 3.2 in the last step. Estimates (4.7) and (4.8) yield together with (4.3) and the embedding $C^{0,\sigma}(\bar{\Omega}) \hookrightarrow L^\infty(\Omega)$ the assertion (4.1).

Since $P_h\bar{u} - P_hR_h\bar{u} = S_h^*S_h\bar{u} - S_h^*S_hR_h\bar{u}$ inequality (4.2) results from (4.1) and the fact that S_h^* is bounded from U to U (see Lemma 3.2). □

Lemma 4.2. *The inequality*

$$\nu\|R_h\bar{u} - \bar{u}_h\|_U^2 \leq (R_h\bar{w} - \bar{w}_h, \bar{u}_h - R_h\bar{u})_U$$

holds.

Proof. A proof of this lemma is given in [35]. The assertion can be derived from the optimality condition (2.3) and the proof is independent of the underlying discretization. □

Now we are able to state the following supercloseness result.

Theorem 4.3. *The inequality*

$$\|\bar{u}_h - R_h\bar{u}\|_U \leq ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right)$$

is valid.

Proof. This result follows with assumption (A9) like in the proof of Theorem 4.21 in [37]. For the sake of completeness we sketch the proof here. From Lemma 4.2 we have

$$\begin{aligned}\nu\|\bar{u}_h - R_h\bar{u}\|_U^2 &\le (R_h\bar{w} - \bar{w}_h, \bar{u}_h - R_h\bar{u})_U \\ &= (R_h\bar{w} - \bar{w}, \bar{u}_h - R_h\bar{u})_U + (\bar{w} - P_hR_h\bar{u}, \bar{u}_h - R_h\bar{u})_U \\ &\quad + (P_hR_h\bar{u} - \bar{w}_h, \bar{u}_h - R_h\bar{u})_U. \end{aligned} \tag{4.9}$$

We estimate these three terms separately. For the first term, we use that Q_h is an L^2-projection and get

$$\begin{aligned}(R_h\bar{w} - \bar{w}, \bar{u}_h - R_h\bar{u})_U &= (R_h\bar{w} - Q_h\bar{w}, \bar{u}_h - R_h\bar{u})_U + (Q_h\bar{w} - \bar{w}, \bar{u}_h - R_h\bar{u})_U \\ &= (R_h\bar{w} - Q_h\bar{w}, \bar{u}_h - R_h\bar{u})_U.\end{aligned}$$

The Cauchy-Schwarz inequality yields together with assumption (A9)

$$\begin{aligned}(R_h\bar{w} - \bar{w}, \bar{u}_h - R_h\bar{u})_U &\le \|R_h\bar{w} - Q_h\bar{w}\|_U\|\bar{u}_h - R_h\bar{u}\|_U \\ &\le ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right)\|\bar{u}_h - R_h\bar{u}\|_U.\end{aligned} \tag{4.10}$$

For the second term we apply again the Cauchy-Schwarz inequality and use $\bar{w} = P\bar{u}$, so that we arrive at

$$(\bar{w} - P_hR_h\bar{u}, \bar{u}_h - R_h\bar{u})_U \le \|P\bar{u} - P_hR_h\bar{u}\|_U\|\bar{u}_h - R_h\bar{u}\|_U.$$

With the estimates (3.5), (4.2) and the embedding $C^{0,\sigma}(\Omega)^d \hookrightarrow U$, one can conclude

$$\begin{aligned}\|P\bar{u} - P_hR_h\bar{u}\|_U &\le \|P\bar{u} - P_h\bar{u}\|_U + \|P_h\bar{u} - P_hR_h\bar{u}\|_U \\ &\le ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right),\end{aligned}$$

and therefore

$$(\bar{w} - P_hR_h\bar{u}, u_h - R_h\bar{u})_U \le ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right)\|\bar{u}_h - R_h\bar{u}\|_U. \tag{4.11}$$

The third term can simply be omitted since

$$\begin{aligned}(P_hR_h\bar{u} - \bar{w}_h, \bar{u}_h - R_h\bar{u})_U &= (P_hR_h\bar{u} - P_h\bar{u}_h, \bar{u}_h - R_h\bar{u})_U \\ &= (S_h(R_h\bar{u} - \bar{u}_h), S_h(\bar{u}_h - R_h\bar{u}))_U \\ &\le 0.\end{aligned} \tag{4.12}$$

Thus, one can conclude from the estimates (4.9)–(4.12)

$$\nu\|\bar{u}_h - R_h\bar{u}\|_U^2 \le ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{o,\sigma}(\Omega)^d}\right)\|u_h - R_h\bar{u}\|_U$$

what yields the assertion. ☐

We compute an approximate control in a post-processing step. To this end the control $\tilde{u}_h$ is constructed as projection of the approximate adjoint velocity $\bar{w}_h$ in the set of admissible controls,

$$\tilde{u}_h = \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{w}_h\right).$$

In the following theorem we formulate our main result.

Theorem 4.4. *Assume that the assumptions (A1)–(A10) hold. Then the estimates*

$$\|\bar{v} - \bar{v}_h\|_U \leq ch^2 \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right), \tag{4.13}$$

$$\|\bar{w} - \bar{w}_h\|_U \leq ch^2 \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right), \tag{4.14}$$

$$\|\bar{u} - \tilde{u}_h\|_U \leq ch^2 \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right) \tag{4.15}$$

are valid with a positive constant c independent of h.

Proof. In order to prove the first assertion we apply the triangle inequality and get

$$\begin{aligned}\|\bar{v} - \bar{v}_h\|_U &= \|S\bar{u} - S_h\bar{u}_h\|_U \\ &\leq \|Su - S_hu\|_U + \|S_h\bar{u} - S_hR_h\bar{u}\|_U + \|S_h(R_h\bar{u} - \bar{u}_h)\|_U.\end{aligned}$$

The first term is a finite element error and is estimated in (3.2). For the second term we consider inequality (4.1). For the third term we use the supercloseness result of Theorem 4.3 and the boundedness of S_h given in Lemma 3.2. These three estimates together yield (4.13). In a similar way one can prove inequality (4.14) by using the estimates (3.5) and (4.2) and again Theorem 4.3 and Lemma 3.2. By using the Lipschitz continuity of the projection operator, we get

$$\|\bar{u} - \tilde{u}_h\|_U = \left\|\Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{w}\right) - \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{w}_h\right)\right\|_U \leq \frac{1}{\nu}\|\bar{w} - \bar{w}_h\|_U$$

and inequality (4.15) is a direct consequence of estimate (4.14). □

5 Example in 3D with anisotropic mesh grading and non-conforming discretization

In this section we consider the optimal control problem (1.1)–(1.3) in a prismatic domain with a reentrant edge. We show, that the assumptions (A1)–(A10) are fulfilled for an approximation of the velocity in the Crouzeix-Raviart finite element space and an approximation of the pressure in the space of piecewise constant functions. We set $\Omega = G \times Z$, where $G \subset \mathbb{R}^2$ is a bounded polygonal domain and $Z := (0, z_0) \subset \mathbb{R}$ is an interval. Additionally, it is assumed that the cross-section G has only one reentrant corner located at the origin. Thus Ω has one reentrant edge $\mathcal{E}$ which is part of the x_3-axis. This is no restriction since the case of more than one reentrant corner in G can be reduced to this situation by a localization argument.

5.1 Regularity

The regularity of the solution of the Stokes equation (1.2) in the prismatic domain Ω introduced above, can be expressed in weighted Sobolev spaces. We denote by $r(x)$ the distance of x to the singular edge and define

$$V_\beta^{k,p}(\Omega) := \left\{ v \in D'(\Omega) : \|v\|_{V_\beta^{k,p}(\Omega)} < \infty \right\}$$

with

$$\|v\|_{V_\beta^{k,p}(\Omega)} := \left(\int_\Omega \sum_{|\alpha| \le k} r^{p(\beta - k + |\alpha|)} |D^\alpha v|^p \, \mathrm{d}x \right)^{1/p}.$$

Here, $k \in \mathbb{N}$, $p \in [1, \infty)$ and $\beta \in \mathbb{R}$ is assumed. The spaces $V_\beta^{k,2}(\Omega)$ will play the role of the spaces $H_\omega^k(\Omega)$ introduced in Section 2.

We recall some regularity results for the Stokes equations in these spaces.

Lemma 5.1. *Assume that* $u \in L^p(\Omega)^3$, $\frac{6}{5} \le p < \infty$ *and let* $\lambda > 0$ *be the smallest positive solution of*

$$\sin(\lambda\omega) = -\lambda \sin \omega \tag{5.1}$$

where ω *is the interior angle at the edge. Then the solution* $(v, p) \in X \times M$ *of the Stokes problem* (1.2) *satisfies*

$$v \in V_\beta^{2,p}(\Omega)^3 \text{ and } p \in V_\beta^{1,p}(\Omega) \quad \forall \beta > 2 - \lambda - \frac{2}{p} \tag{5.2}$$

and the a priori estimate

$$\|v\|_{V_\beta^{2,p}(\Omega)^3} + \|p\|_{V_\beta^{1,p}(\Omega)} \le c\|u\|_{L^p(\Omega)} \tag{5.3}$$

holds. Further one has for $u \in L^2(\Omega)^3$

$$\partial_3 v \in V_0^{1,2}(\Omega)^3 \quad \text{and} \quad \partial_3 p \in L^2(\Omega) \tag{5.4}$$

with the corresponding a priori estimate

$$\|\partial_3 v\|_{V_0^{1,2}(\Omega)^3} + \|\partial_3 p\|_{L^2(\Omega)} \le c\|u\|_{L^2(\Omega)}. \tag{5.5}$$

Proof. The assertions (5.2) and (5.3) follow from Theorem 6.1 of [27]. In our case for the vertex eigenvalues λ_q the inequality $\mathrm{Re}\lambda_q \ge 1$ holds [31]. This means we can choose $\beta_q = 0$ in Theorem 6.1 of [27]. So setting $m = 2$ in this theorem yields (5.2) and (5.3). The extra regularity in edge direction stated in (5.4) and (5.5) is proved in Theorem 2.1 of [3]. □

Remark 5.2. *The smallest positive solution* λ *of* (5.1) *satisfies*

$$\frac{1}{2} < \lambda < \frac{\pi}{\omega},$$

see e.g. [16].

Corollary 5.3. *For the solution* $(\bar{u}, \bar{v}, \bar{p})$ *of the optimal control problem* (1.1)–(1.3) *one has for* $\sigma \in \left(0, \frac{1}{2}\right)$

$$\bar{v} \in C^{0,\sigma}(\bar{\Omega})^3 \text{ and } \bar{u} \in C^{0,\sigma}(\bar{\Omega})^3 \tag{5.6}$$

and further

$$\|\bar{v}\|_{L^\infty(\Omega)^3} \leq \|\bar{v}\|_{C^{0,\sigma}(\bar{\Omega})^3} \leq c\|\bar{u}\|_{L^\infty(\Omega)^3} \leq c\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^3}. \tag{5.7}$$

Proof. In the following we utilize Lemma 5.1 and the projection formula (2.4). Due to the definition of the problem, one has $\bar{u} \in L^2(\Omega)^3$. This means for a value $\mu < \lambda$ that $\bar{v} \in V^{2,2}_{1-\mu}(\Omega)^3$ since $1 - \mu > 1 - \lambda$. Since $\lambda > 1/2$ (see Remark 5.2) one can always choose a value for μ such that $1/2 < \mu < \lambda$. Then the embedding $V^{2,2}_{1-\mu}(\Omega) \hookrightarrow V_0^{2-(1-\mu),2}(\Omega) \hookrightarrow W^{1+\mu,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ holds according to [36, Lemma 1.2], the Sobolev embedding theorem and the fact that $1 + \mu - 3/2 > 0$. This yields $\bar{v} \in L^\infty(\Omega)^3$ and therefore $\bar{v} - v_d \in L^\infty(\Omega)^3$. Applying Lemma 5.1 to the adjoint equation yields $\bar{w} \in V^{2,p}_\beta(\Omega)^3$ for all $p \geq 6/5$ and $\beta > 2 - \lambda - 2/p$. In the following we choose β such that

$$2 - \lambda - \frac{2}{p} < \beta < 2 - \frac{3}{p} - \sigma.$$

This is possible as long as $\lambda > 1/p + \sigma$. Since $\sigma < 1/2$ and $\lambda > 1/2$ this can be guaranteed for p large enough. With this setting the embedding $V^{2,p}_\beta(\Omega) \hookrightarrow V_0^{2-\beta,p}(\Omega) \hookrightarrow W^{2-\beta,p}(\Omega) \hookrightarrow C^{0,\sigma}(\bar{\Omega})$ holds and it follows $\bar{w} \in C^{0,\sigma}(\bar{\Omega})^3$. The projection formula (2.4) yields $\bar{u} \in C^{0,\sigma}(\bar{\Omega})^3$. With the same argumentation for the state equation one can conclude $\bar{v} \in C^{0,\sigma}(\bar{\Omega})^3$. The estimate (5.7) follows then from Lemma 5.1,

$$\|\bar{v}\|_{L^\infty(\Omega)^3} \leq \|\bar{v}\|_{C^{0,\sigma}(\bar{\Omega})^3} \leq c\|\bar{v}\|_{V^{2,p}_\beta(\Omega)^3} \leq c\|\bar{u}\|_{L^p(\Omega)^3} \leq c\|\bar{u}\|_{L^\infty(\Omega)^3} \leq c\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^3}$$

where we have chosen p large enough. □

Corollary 5.4. *For the optimal adjoint velocity* $\bar{w}$ *one has* $\bar{w} \in W^{1,p}(\Omega)^3$ *and*

$$\|\bar{w}\|_{W^{1,p}(\Omega)^3} \leq c\left(\|\bar{u}\|_{L^\infty(\Omega)^3} + \|v_d\|_{L^\infty(\Omega)^3}\right) \tag{5.8}$$

with $p < \frac{2}{1-\lambda}$. *Furthermore it is* $\bar{w} \in V^{1,p}_{1-\mu}(\Omega)$ *and*

$$\|\bar{w}\|_{V^{1,p}_{1-\mu}(\Omega)^3} \leq c\left(\|\bar{u}\|_{L^\infty(\Omega)^3} + \|v_d\|_{L^\infty(\Omega)^3}\right) \tag{5.9}$$

for all $p > 1$ *and* $\mu < \lambda + \frac{2}{p}$.

Proof. Since $p < \frac{2}{1-\lambda}$ it is $1 > 2 - \lambda - \frac{2}{p}$. Therefore we can choose $\beta = 1$ in Lemma 5.1. Since $V_1^{2,p}(\Omega)^3 \hookrightarrow V_0^{1,p}(\Omega)^3 \hookrightarrow W^{1,p}(\Omega)^3$ it follows from (5.2) and (5.3), that

$$\|\bar{w}\|_{W^{1,p}(\Omega)^3} \le c\|\bar{v} - v_d\|_{L^p(\Omega)^3}.$$

Then the assertion (5.8) follows from the embedding $L^\infty(\Omega) \hookrightarrow L^p(\Omega)$ and Corollary 5.3. For the proof of (5.9) we set $\beta = 2 - \mu$ in (5.3). This is possible due to the fact that $2 - \mu > 2 - \lambda - \frac{2}{p}$ for all $p > 0$ since $\mu < \lambda$. With embedding $V_{2-\mu}^{2,p}(\Omega) \hookrightarrow V_{1-\mu}^{1,p}(\Omega)$ we can write

$$\|\bar{w}\|_{V_{1-\mu}^{1,p}(\Omega)} \le c\|\bar{v} - v_d\|_{L^p(\Omega)}.$$

This yields together with the triangle inequality, Corollary 5.3 and the embedding $L^\infty(\Omega) \hookrightarrow L^p(\Omega)$ the assertion (5.9). □

Lemma 5.5. *Let $v_d \in C^{0,\sigma}(\bar{\Omega})$, $\sigma \in (0, 1/2)$, and $\gamma > 1 - \lambda$. Then the inequality*

$$\|r^\gamma \nabla P\bar{u}\|_{L^\infty(\Omega)^3} \le c\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^3} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^3}\right)$$

is valid.

Proof. In order to prove the assertion we utilize Theorem 6.1 of [28]. We set $l = 2$ and $\delta = \beta$ in that Theorem. This results in the condition $2 - \lambda < \delta - \sigma < 2$, what is equivalent to

$$1 - \lambda < \delta - \sigma - 1 < 1. \tag{5.10}$$

Since $\bar{v} - v_d = S\bar{u} - v_d \in C^{0,\sigma}(\bar{\Omega})$ (comp. Corollary 5.3) we can conclude $Pu \in C_{\delta,\delta}^{2,\sigma}(\Omega)^3$ for δ satisfying (5.10). The definition of this weighted Hölder space is given on page 1013 of [28]. Taking this definition into account, one can conclude

$$r^{\delta-1-\sigma}\nabla Pu \in L^\infty(\Omega).$$

If we set $\gamma = \delta - \sigma - 1$, it finally follows

$$\|r^\gamma \nabla Pu\|_{L^\infty(\Omega)} \le c\|\bar{v} - v_d\|_{C^{0,\sigma}(\Omega)} \text{ for } \gamma > 1 - \lambda.$$

The application of the triangle inequality and (5.7) yields the assertion. □

5.2 Discretization

We define a family of anisotropic, graded meshes $\mathcal{T}_h = \{T\}$ which satisfy assumption (A1). First, we introduce a graded, isotropic triangulation $\{\tau\}$ in the two-dimensional domain G. The elements are triangles. With h being the global mesh parameter, $\mu \in (0, 1]$ being the grading parameter and r_τ being the distance to the corner,

$$r_\tau := \inf_{(x_1,x_2)\in\tau} (x_1^2 + x_2^2)^{1/2}$$

the element size $h_\tau = \operatorname{diam} \tau$ is assumed to satisfy

$$h_\tau \sim \begin{cases} h^{1/\mu} & \text{for } r_\tau = 0, \\ h r_\tau^{1-\mu} & \text{for } 0 < r_\tau \le R, \\ h & \text{for } r_\tau > R. \end{cases}$$

Here, R is some constant. From this graded two-dimensional mesh we build a three-dimensional mesh of pentahedra by extruding the triangles τ in x_3-direction with uniform mesh size h. In order to generate an anisotropic graded tetrahedral mesh, we divide each of these pentahedra into tetrahedra. Note that the number of elements is of order h^{-3} for every $\mu \in (0, 1]$. We can characterize the elements T of such a mesh by the three sizes $h_{T,1}$, $h_{T,2}$ and $h_{T,3}$, where $h_{T,i}$ is the length of projection of T on the x_i-axis, $i = 1, 2, 3$. In detail, with r_T being the distance of the element T to the edge,

$$r_T := \inf_{x \in T} (x_1^2 + x_2^2)^{1/2},$$

the element sizes satisfy

$$\begin{aligned} h_{T,i} &\sim h^{1/\mu} && \text{for } r_T = 0, \\ h_{T,i} &\sim h r_T^{1-\mu} && \text{for } r_T > 0, \\ h_{T,3} &\sim h, \end{aligned} \tag{5.11}$$

for $i = 1, 2$. For the grading parameter, we demand

$$\mu < \lambda.$$

We approximate the velocity by Crouzeix-Raviart elements,

$$X_h := \left\{ v_h \in L^2(\Omega)^3 : v_h|_T \in (\mathcal{P}_1)^3 \ \forall T, \int_F [v_h]_F = 0 \ \forall F \right\}$$

where F denotes a face of an element and $[v_h]_F$ means the jump of v_h on the face F,

$$[v_h(x)]_F := \begin{cases} \lim_{\alpha \to 0} (v_h(x + \alpha n_F) - v_h(x - \alpha n_F)) & \text{for an interior face F}, \\ v_h(x) & \text{for a boundary face F}. \end{cases}$$

Here n_F is the outer normal of F. For the approximation of the pressure we use piecewise constant functions, this means

$$M_h := \left\{ q_h \in L^2(\Omega) : q_h|_T \in \mathcal{P}_0 \ \forall T, \int_\Omega q_h = 0 \right\}.$$

5.3 Proof of assumptions (A2)–(A10)

The spaces $V_\beta^{k,2}(\Omega)$ play the role of the spaces $H_\omega^k(\Omega)$. The corresponding regularity results, that prove assumption (A2), are given in Lemma 5.1.

The discrete Poincaré inequality of assumption (A3),

$$\|v_h\|_{L^2(\Omega)} \le c\|v_h\|_{X_h} \quad \forall v_h \in X_h,$$

is proved in [25, Corollary 5.4].

The assumption (A4) follows from the embedding

$$V_{1-\mu}^{2,2}(\Omega) \hookrightarrow V_0^{2-(1-\mu),2}(\Omega) \hookrightarrow W^{1+\mu,2}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

The first embedding is proved in [36]. The second one follows directly from the definition of the spaces. The last embedding is a conclusion from the Sobolev embedding theorem and the fact that one can choose μ such that $1/2 < \mu < \lambda$ and therefore $1 + \mu - \frac{3}{2} > 0$.

In order to prove (A5) we introduce the interpolant E_{0h} defined by

$$(E_{0h}v)(x) := \sum_{i\in I} a_i\varphi_i(x).$$

Here, I denotes the index set of all nodes X_i not belonging to $G \times \{0, z_0\}$. The functions φ_i are the nodal basis functions. The coefficients a_i are defined as value of the $L^2(\sigma_i)$-projection of v into the space of constants over σ_i. The subset σ_i is chosen such that it satisfies the following conditions.

- (P1) σ_i is one-dimensional and parallel to the x_3-axis.
- (P2) $X_i \in \overline{\sigma}_i$
- (P3) There exists an edge e of some element T such that the projection of e on the x_3-axis coincides with the projection of σ_i.
- (P4) If the projections of any two points X_i and X_j on the x_3-axis coincide then so do the projections of σ_i and σ_j.

This interpolant was originally introduced in [6]. It is a modified version of the quasi-interpolant E_h from [1] and is chosen such that homogeneous Dirichlet boundary conditions are preserved. In the following we choose $i_h^v := E_{0h}$. Notice that we do not use the Crouzeix-Raviart interpolant although we use Crouzeix-Raviart elements for the velocity and although the estimates in (A5) could be fulfilled by this interpolant. The reason for this is that the Crouzeix-Raviart interpolant maps to X_h but not to $X \cap X_h$ as demanded in (A5).

Proof of (A5). The estimate (A5)(i) is proved in Theorem 5.1 of [6]. In order to prove (A5)(ii) we write

$$E_{0h}v(x) = \sum_{i\in I}\left[\frac{1}{|\sigma_i|}\int_{\sigma_i} v\right]\varphi_i(x).$$

Since v is as solution of (1.2) a continuous function, we can conclude

$$\|E_{0h}v\|_{L^\infty(\Omega)} = \sup_{i\in I}\left|\frac{1}{|\sigma_i|}\int_{\sigma_i} v\right| \le \|v\|_{L^\infty(\bar\Omega)}. \tag{5.12}$$

The embedding $V^{2,2}_\beta(\Omega) \hookrightarrow L^\infty(\Omega)$ yields together with Lemma 5.1

$$\|v\|_{L^\infty(\bar\Omega)} \le c\|v\|_{V^{2,2}_\beta(\Omega)} \le \|u\|_{L^2(\Omega)}. \tag{5.13}$$

Now we can conclude with the triangle inequality and the estimates (5.12) and (5.13)

$$\|v - E_{0h}v\|_{L^\infty(\Omega)} \le \|v\|_{L^\infty(\Omega)} + \|E_{0h}v\|_{L^\infty(\Omega)} \le \|u\|_{L^2(\Omega)}$$

and (A5)(ii) is proved. To prove (A5)(iii) we set $i_h^p := Q_h$ as defined in (2.10). The assertion is shown in the proof of Lemma 3.2 in [3]. Notice, that M_h in that proof is the interpolant Q_h in our setting. □

In the following we prove assumption (A6).

Proof of (A6). For an arbitrary $\varphi_h \in X_h$ one has

$$\|\varphi_h\|_{L^\infty(T)} = \|\hat\varphi_h\|_{L^\infty(\hat T)} \le c\|\hat\varphi_h\|_{L^p(\hat T)} \le c|T|^{-1/p}\|\varphi_h\|_{L^p(T)}.$$

We choose the minimal element size according to (5.11) and arrive at

$$\|\varphi_h\|_{L^\infty(\Omega)} \le ch^{-2/(\mu p)}h^{-1/p}\|\varphi_h\|_{L^p(\Omega)}.$$

In order to achieve $-\frac{2}{\mu p} - \frac{1}{p} \ge -1$ one has to demand $p \ge \frac{2}{\mu} + 1$. This condition is no contradiction to $p \le 6$ as long as $\mu \ge \frac{2}{5}$. But this can be satisfied since μ has only to fulfill the condition $\mu < \lambda$ and $\lambda > \frac{1}{2}$ (comp. Remark 5.2). Therefore it exists $p \in \left[\frac{2}{\mu} + 1, 6\right]$ such that

$$\|\varphi_h\|_{L^\infty(\Omega)} \le ch^{-1}\|\varphi_h\|_{L^p(\Omega)}.$$

what is the inequality of assumption (A6). □

The consistency error estimate (A7) and the discrete inf-sup-condition (A8) are proved in [3].

In [37] assumptions (A9) and (A10) are proved for the solution of the optimal control problem (1.1) governed by the Poisson equation. In their proof the authors made use of results concerning the regularity of the solution along the edge in the spaces $L^p(\Omega)$ for general p. Since such estimates are not available for the Stokes equation, a component-wise consideration of their arguments is not possible here. In the following we prove assumptions (A9) and (A10) without using additional regularity along the edge in the spaces $L^p(\Omega)$ for $p \neq 2$.

First of all we recall two lemmata from [37] concerning the projection operators R_h and Q_h.

Lemma 5.6. *Let $\mathcal{T}_h$ be a conforming anisotropic triangulation satisfying* (5.11) *and let R_h be the projection defined in* (2.9). *Then there holds*

$$\left|\int_T (f - R_h f)\,\mathrm{d}x\right| \leq \begin{cases} c|T|^{1/2} \sum_{|\alpha|=2} h_T^\alpha \|D^\alpha f\|_{L^2(T)} & \text{for } f \in H^2(T) \\ c|T| \sum_{|\alpha|=1} h_T^\alpha \|D^\alpha f\|_{L^\infty(T)} & \text{for } f \in W^{1,\infty}(T) \\ c|T| \|f\|_{C(\bar{T})} & \text{for } f \in C(\bar{T}). \end{cases}$$

Lemma 5.7. *Let the mesh be graded according to* (5.11). *Then the projection operators R_h and Q_h defined in* (2.9) *and* (2.10) *satisfy the inequality*

$$\|Q_h f - R_h f\|_{L^2(T)} \leq |T|^{1/2-1/p} \sum_{|\alpha|=1} h^\alpha \|D^\alpha f\|_{L^p(T)}$$

for all $f \in W^{1,p}(T)$ with $p > 3$.

Now we are ready to prove (A9).

Proof of (A9). First of all we split Ω in two parts,

$$K_s = \bigcup_{\bar{T} \cap \mathcal{E} \neq \emptyset} T, \qquad K_r = \Omega \backslash \bar{K}_s. \tag{5.14}$$

First we prove the estimate in K_r. Notice, that one has $\bar{w} \in H^2(K_r)^3$. We write for each component $\bar{w}_k$ $(k = 1, 2, 3)$ of $\bar{w} = (\bar{w}_1, \bar{w}_2, \bar{w}_3)$

$$\begin{aligned} \|Q_h \bar{w}_k - R_h \bar{w}_k\|^2_{L^2(K_r)} &= \sum_{T \subset K_r} \|Q_h \bar{w}_k - R_h \bar{w}_k\|^2_{L^2(T)} \\ &= \sum_{T \subset K_r} |T|^{-1} \left|\int_T (\bar{w}_k - R_h \bar{w}_k)\,\mathrm{d}x\right|^2. \end{aligned}$$

Now we can apply Lemma 5.6 and get

$$\begin{aligned}\|Q_h\bar w_k - R_h\bar w_k\|^2_{L^2(K_r)} &\le \sum_{T\subset K_r} |T|^{-1}\left[c|T|^{1/2}\sum_{|\alpha|=2} h_T^\alpha \|D^\alpha \bar w_k\|_{L^2(T)}\right]^2 \\ &\le c\sum_{T\subset K_r}\left[\sum_{|\alpha|=2} h_T^\alpha\|D^\alpha \bar w_k\|_{L^2(T)}\right]^2 \\ &\le c\sum_{T\subset K_r}\left[ch^2\left(\sum_{i=1}^2\sum_{j=1}^2 \|r^{2-2\mu}\partial_{ij}\bar w_k\|_{L^2(T)}+\right.\right. \\ &\qquad\left.\left.\sum_{i=1}^2\|r^{1-\mu}\partial_{3i}\bar w_k\|_{L^2(T)} + \|\partial_{33}\bar w_k\|_{L^2(T)}\right)\right]^2 \\ &\le ch^4\left(|\bar w_k|^2_{V^{2,2}_{2-2\mu}(K_r)} + |\partial_3\bar w_k|^2_{V^{1,2}_{1-\mu}(K_r)} + |\partial_{33}\bar w_k|^2_{V^{0,2}_0(K_r)}\right).\end{aligned}$$

This yields

$$\|Q_h\bar w - R_h\bar w\|_{L^2(K_r)^3} \le ch^2\left(|\bar w|^2_{V^{2,2}_{2-2\mu}(K_r)^3} + |\partial_3\bar w|^2_{V^{1,2}_{1-\mu}(K_r)^3} + |\partial_{33}\bar w|^2_{V^{0,2}_0(K_r)^3}\right)^{1/2}.$$

With the a priori estimates of Lemma 5.1, the embedding $C^{0,\sigma}(\bar\Omega)\hookrightarrow L^2(\Omega)$ and Corollary 5.3 one gets

$$\|Q_h\bar w - R_h\bar w\|_{L^2(K_r)^3} \le ch^2\|v-v_d\|_{L^2(\Omega)} \le ch^2\left(\|\bar u\|_{C^{0,\sigma}(\bar\Omega)^3} + \|\bar v_d\|_{C^{0,\sigma}(\bar\Omega)^3}\right). \quad (5.15)$$

We proceed with the estimate in the subdomain K_s. We choose p and γ such that

$$p>3,\quad p<\frac{2}{1-\lambda},\quad p<\frac{2}{\gamma},\quad \gamma<1-\mu \text{ and } \gamma>1-\lambda. \quad (5.16)$$

Since $\lambda>\frac12$ one has $3<\frac{2}{1-\lambda}$. Further it is $3<\frac{2}{\gamma}$ if $\gamma<\frac23$. This can be fulfilled since $\frac23>1-\lambda$. Finally $1-\lambda<1-\mu$ due to the fact that $\mu<\lambda$. Altogether this means, that there are actually p and γ that satisfy the assumptions in (5.16).

¿From Corollary 5.4 one has $\bar w\in W^{1,p}(\Omega)^3$. Now we can apply Lemma 5.7 on every component $\bar w_k$, $(k=1,2,3)$ of $\bar w$ and conclude

$$\begin{aligned}\|Q_h\bar w_k - R_h\bar w_k\|^2_{L^2(K_s)} &= \sum_{T\subset K_s}\|Q_h\bar w_k - R_h\bar w_k\|^2_{L^2(T)} \\ &\le c\sum_{T\in K_s}|T|^{1-2/p}\left(\sum_{|\alpha|=1} h_T^\alpha\|D^\alpha\bar w_k\|_{L^p(T)}\right)^2 \\ &\le c\sum_{|\alpha|=1}\sum_{T\in K_s}|T|^{1-2/p}h_T^{2\alpha}\|D^\alpha\bar w_k\|^2_{L^p(T)}\end{aligned}$$

Since $h_T^{2\alpha} \le ch^2$ for all $|\alpha| = 1$, one can continue with Lemma 5.5

$$\begin{aligned}\|Q_h\bar{w}_k - R_h\bar{w}_k\|^2_{L^2(K_s)} &\le ch^2 \sum_{T\in K_s} |T|^{1-2/p}\|r^{-\gamma}r^{\gamma}\nabla\bar{w}_k\|^2_{L^p(T)} \\ &\le ch^2\|r^{\gamma}\nabla\bar{w}_k\|^2_{L^\infty(\Omega)} \sum_{T\subset K_s} |T|^{1-2/p}\|r^{-\gamma}\|^2_{L^p(T)}. \end{aligned} \tag{5.17}$$

In the following we prove that the inequality

$$\sum_{T\subset K_s} |T|^{1-2/p}\|r^{-\gamma}\|^2_{L^p(T)} \le ch^2$$

is valid. To this end we apply the Hölder inequality and get

$$\begin{aligned}\sum_{T\subset K_s} |T|^{1-2/p}\|r^{-\gamma}\|^2_{L^p(T)} &\le \left[\sum_{T\subset K_s}\left(|T|^{1-2/p}\right)^{\frac{p}{p-2}}\right]^{\frac{p-2}{p}} \left[\sum_{T\subset K_s}\|r^{-\gamma}\|^p_{L^p(T)}\right]^{\frac{2}{p}} \\ &\le c\left(\sum_{T\subset K_s}|T|\right)^{\frac{p-2}{p}}\left(\int_0^{h^{1/\mu}} r^{-\gamma p} r\,\mathrm{d}r\right)^{\frac{2}{p}} \\ &\le c|K_s|^{\frac{p-2}{p}}\left(h^{\frac{1}{\mu}(2-\gamma p)}\right)^{\frac{2}{p}}\end{aligned}$$

where we have used $\gamma < \frac{2}{p}$ (comp. (5.16)) in the last step. Because $|K_s| \le ch^{2/\mu}$ one can conclude

$$\sum_{T\subset K_s} |T|^{1-2/p}\|r^{-\gamma}\|^2_{L^p(T)} \le ch^{\frac{2}{\mu}\left(\frac{p-2}{p}+\frac{2-\gamma p}{p}\right)} = ch^{\frac{2}{\mu}(1-\gamma)} \le ch^2.$$

using the fact that $\mu < 1-\gamma$. This estimate yields together with (5.17) the inequality

$$\|Q_h\bar{w} - R_h\bar{w}\|_{L^2(K_s)^3} \le ch^2\|r^{\gamma}\nabla\bar{w}\|_{L^\infty(\Omega)^3} \tag{5.18}$$

and with Lemma 5.5

$$\|Q_h\bar{w} - R_h\bar{w}\|_{L^2(K_s)^3} \le ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^3} + \|\bar{v}_d\|_{C^{0,\sigma}(\bar{\Omega})^3}\right). \tag{5.19}$$

Together with estimate (5.15) this yields assumption (A9). □

It remains to prove assumption (A10).

Due to Lemma 2.1 the optimal control $\bar{u}$ results from the projection of the adjoint velocity $\bar{w}$ to the admissible set U^{ad}. Since this projection may cause some kinks in $\bar{u}$, the control can be less regular than the adjoint velocity. Therefore we classify the elements $T \in \mathcal{T}_h$ in two sets,

$$K_1 := \bigcup_{T\in\mathcal{T}_h:\bar{u}\notin V^{2,2}_\beta(T)^3} T, \qquad K_2 := \bigcup_{T\in\mathcal{T}_h:\bar{u}\in V^{2,2}_\beta(T)^d} T.$$

Although the number of elements in K_1 grows for decreasing h, the assumption

$$\#K_1 \le ch^{-2} \tag{5.20}$$

is fulfilled in many practical cases. A detailed discussion on this assumption can be found in [37].

In [7] it is proved that (A10) is valid for the Poisson equation under the assumptions $\mu < \lambda$ and (5.20). The regularity of the adjoint state plays a crucial role in that proof. Since the regularity properties of each component of the velocity field of the Stokes problem are similar to that of the Poisson equation, the following proof is also similar to that given in [7]. The result for the Stokes equation follows by a componentwise consideration. For the sake of completeness we sketch the proof here.

Proof of (A10). We split the domain Ω in three parts, $K_{1,r} = K_1 \backslash \bar{K}_s$, $K_{2,r} = K_2 \backslash \bar{K}_s$ and K_s as defined in the proof of (A9). Again we estimate each component of $\bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$ separately. We get for $k = 1, 2, 3$ and $\varphi_h = (\varphi_{h,1}, \varphi_{h,2}, \varphi_{h,3}) \in X_h$

$$\left| \int_\Omega \varphi_{h,k} (Q_h \bar{u}_k - R_h \bar{u}_k) \, \mathrm{d}x \right| \le \sum_{T \in \mathcal{T}_h} \|\varphi_{h,k}\|_{L^\infty(T)} \left| \int_T (\bar{u}_k - R_h \bar{u}_k) \, \mathrm{d}x \right|.$$

The application of Lemma 5.6 on each sub-domain to the integral yields

$$\begin{aligned} \left| \int_\Omega \varphi_{h,k} (Q_h \bar{u}_k - R_h \bar{u}_k) \, \mathrm{d}x \right| \le & \sum_{T \subset K_{2,r}} \|\varphi_{h,k}\|_{L^\infty(T)} |T|^{1/2} \sum_{|\alpha|=2} h_T^\alpha \|D^\alpha \bar{u}_k\|_{L^2(T)} \\ & + \sum_{T \subset K_{1,r}} \|\varphi_{h,k}\|_{L^\infty(T)} |T| \sum_{|\alpha|=1} h_T^\alpha \|D^\alpha \bar{u}_k\|_{L^\infty(T)} \\ & + \sum_{T \subset K_s} \|\varphi_{h,k}\|_{L^\infty(T)} |T| \|\bar{u}_k\|_{L^\infty(T)}. \end{aligned} \tag{5.21}$$

Analogously to the proof given in [7] we continue with estimating the three terms separately. Using (5.11), we get for the first term of the right-hand side of inequality (5.21)

$$\sum_{T \subset K_{2,r}} \|\varphi_{h,k}\|_{L^\infty(T)} |T|^{1/2} \sum_{|\alpha|=2} h_T^\alpha \|D^\alpha \bar{u}_k\|_{L^2(T)} \tag{5.22}$$

$$\begin{aligned} \le & c \|\varphi_{h,k}\|_{L^\infty(K_{2,r})} |K_{2,r}|^{1/2} \left(h^2 \sum_{i=1}^2 \sum_{j=1}^2 \|r^{2-2\mu} \partial_{ij} \bar{u}_k\|_{L^2(K_{2,r})} \right. \\ & \left. + h^2 \sum_{i=1}^2 \|r^{1-\mu} \partial_{3i} \bar{u}_k\|_{L^2(K_{2,r})} + h^2 \|\partial_{33} \bar{u}_k\|_{L^2(K_{2,r})} \right). \end{aligned} \tag{5.23}$$

The second term can be estimated by

$$\begin{aligned}
&\sum_{T\subset K_{1,r}} \|\varphi_{h,k}\|_{L^\infty(T)}|T| \sum_{|\alpha|=1} h_T^\alpha \|D^\alpha \bar u_k\|_{L^\infty(T)} \\
&\qquad \le c\|\varphi_{h,k}\|_{L^\infty(\Omega)} \sum_{T\subset K_{1,r}} |T| \left(h r_T^{1-\mu} \sum_{i=1}^2 \|\partial_i \bar u_k\|_{L^\infty(T)} + h\|\partial_3 \bar u_k\|_{L^\infty(T)} \right) \\
&\qquad \le c\|\varphi_{h,k}\|_{L^\infty(\Omega)} h^4 \sum_{T\subset K_{1,r}} r_T^{2-2\mu} \left(\sum_{i=1}^2 \|r^{1-\mu}\partial_i \bar u_k\|_{L^\infty(T)} + r^{\mu-1}\|r^{1-\mu}\partial_3 \bar u_k\|_{L^\infty(T)} \right) \\
&\qquad \le c\|\varphi_{h,k}\|_{L^\infty(\Omega)} \|r^{1-\mu}\nabla \bar u_k\|_{L^\infty(K_{1,r})} \# K_{1,r} h^4 \\
&\qquad \le ch^2 \|\varphi_{h,k}\|_{L^\infty(\Omega)} \|r^{1-\mu}\nabla \bar u_k\|_{L^\infty(\Omega)} \qquad (5.24)
\end{aligned}$$

In the last step we have used assumption (5.20). Since $|K_s| \le ch^{2/\mu} \le ch^2$ the third term yields

$$\sum_{T\subset K_s} \|\varphi_{h,k}\|_{L^\infty(T)}|T|\|\bar u\|_{L^\infty(T)} \le |K_s|\|\bar u_k\|_{L^\infty(K_s)} \le ch^2\|\bar u_k\|_{L^\infty(K_s)}. \qquad (5.25)$$

Taking the projection formula (2.4) into aaccount we can substitute $\bar u_k$ by $-\frac{1}{\mu}\bar p_k$ because $\bar u_k$ is either constant or equal to $-\frac{1}{\mu}\bar p_k$. Since this is valid for every component, we conclude from (5.23)–(5.25) together with (5.21) the estimate

$$\begin{aligned}
(\varphi_h, Q_h\bar u - R_h\bar u) \le \frac{c}{\nu} h^2 \|\varphi_h\|_{L^\infty(\Omega)^3} \cdot \\
\Bigg(\sum_{i=1}^2\sum_{j=1}^2 \|r^{2-2\mu}\partial_{ij}\bar w\|_{L^2(K_{2,r})^3} + \sum_{i=1}^2 \|r^{1-\mu}\partial_{3i}\bar w\|_{L^2(K_{2,r})^3} + \|\partial_{33}\bar w\|_{L^2(K_{2,r})^3} \qquad (5.26) \\
+ \sum_{i=1}^2 \|r^{1-\mu}\partial_i \bar w\|_{L^\infty(K_{1,r})^3} + \|\partial_3\bar w\|_{L^\infty(K_{1,r})^3} + \nu\|\bar u\|_{L^\infty(K_s)^3} \Bigg). \qquad (5.27)
\end{aligned}$$

The aapplicationof Lemma 5.1 and Corollary 5.5 yields the assertion. □

We have shown, that the assumptions (A1)–(A10) are fulfilled for an approximation of the optimal control problem (1.1)–(1.3) in a prismatic domain with a reentrant edge, where for the velocity Crouzeix-Raviart elements and for pressure and control piecewise constant functions are used. Therefore Theorem 4.4 is valid in this case.

5.4 Numerical tests

In this subsection we illustrate our theoretical findings by a numerical example. In order to be able to construct an analytical solution we consider the slightly modified functional

$$J(v,u) := \frac{1}{2}\|v - v_d\|^2_{L^2(\Omega)^d} + \frac{\nu}{2}\|u\|_{L^2(\Omega)^d} + \int_{\partial\Omega} \frac{\partial v}{\partial n} g \text{ ds}$$

and the state equation

$$\begin{aligned} -\Delta v + \nabla p &= u + f \qquad \text{in } \Omega, \\ \nabla \cdot v &= 0 \qquad \text{in } \Omega, \\ v &= g \qquad \text{on } \partial\Omega. \end{aligned}$$

The adjoint equation is given as

$$\begin{aligned} -\Delta w + \nabla r &= v - v_d \qquad \text{in } \Omega, \\ \nabla \cdot w &= 0 \qquad \text{in } \Omega, \\ w &= g \qquad \text{on } \partial\Omega. \end{aligned}$$

where the inhomogeneous boundary conditions are the result of the last integral term in the functional J. The domain Ω is set as

$$\Omega = \left\{ (r\cos\varphi, r\sin\varphi, x_3) \in \mathbb{R}^3 : 0 < r < 1, 0 < \varphi < \frac{3}{2}\pi, 0 < x_3 < 1 \right\}.$$

The functions f, g and v_d are chosen such that

$$\bar{v} = \bar{w} = \begin{bmatrix} x_3 r^\lambda \Phi_1(\varphi) \\ x_3 r^\lambda \Phi_2(\varphi) \\ r^{2/3} \sin \frac{2}{3}\varphi \end{bmatrix}, \quad \bar{p} = -\bar{r} = x_3 r^{\lambda-1}\Phi_p(\varphi), \quad \bar{u} = \Pi_{[-10,-0.2]}\left(-\frac{1}{\nu}\bar{w}\right)$$

is the exact solution for the optimal control problem. Here, $\lambda \approx 0.5445$ is the smallest positive solution of the eigenvalue problem (5.1). The functions Φ_1, Φ_2 and Φ_p are given as

$$\begin{aligned} \Phi_1(\varphi) =& -\sin(\lambda\varphi)\cos\omega - \lambda\sin(\varphi)\cos(\lambda(\omega-\varphi)+\varphi) \\ &+ \lambda\sin(\omega-\phi)\cos(\lambda\varphi-\varphi) + \sin(\lambda(\omega-\varphi)), \\ \Phi_2(\varphi) =& -\sin(\lambda\varphi)\sin\omega - \lambda\sin(\varphi)\sin(\lambda(\omega-\varphi)+\varphi) \\ &- \lambda\sin(\omega-\varphi)\sin(\lambda\varphi-\varphi), \\ \Phi_p(\varphi) =&\ 2\lambda\left[\sin((\lambda-1)\varphi+\omega) + \sin((\lambda-1)\varphi-\lambda\omega)\right]. \end{aligned}$$

This solution has the typical singular behavior near the edge (comp. [3]).

The problem is solved using a primal-dual active set strategy. For details we refer to [24].

In table 1 one can observe second order convergence in the post-processed control $\tilde{u}_h$ as long as the mesh is sufficiently graded ($\mu = 0.4 < 0.5445 = \lambda$, comp. Fig. 1). This fits our theoretical findings. If one uses a uniform mesh ($\mu = 1.0$), the convergence rate tends to 2λ.

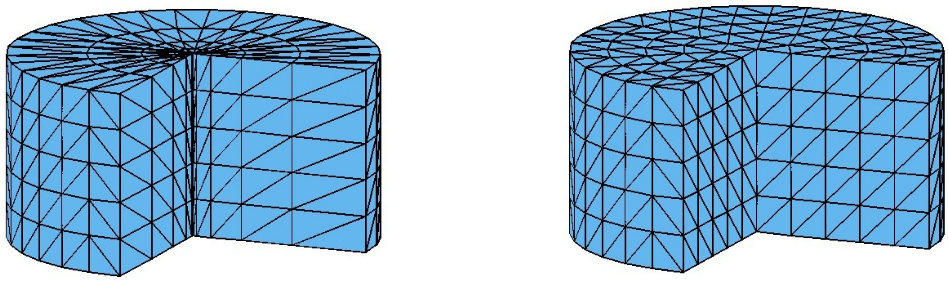

Figure 1: Anisotropic graded mesh with $\mu = 0.4$ (left) and uniform mesh ($\mu = 1.0$)

	$\mu = 0.4$		$\mu = 1$	
ndof	value	rate	value	rate
14025	1.11E-02		1.38E-02	
37779	6.07E-03	1.83	8.86E-03	1.33
108600	3.19E-03	1.83	5.66E-03	1.27
362475	1.49E-03	1.89	3.46E-03	1.23
854400	8.64E-04	1.91	2.45E-03	1.20
1135464	7.20E-04	1.93	2.20E-03	1.18
1663125	5.62E-04	1.94	1.89E-03	1.17

Table 1: L^2-error of the computed control $\tilde{u}_h$ on an anisotropic, three-dimensional mesh

6 Example in 2D with isotropic mesh grading

In this section we consider the optimal control problem (1.1)–(1.3) in a plane domain Ω with a reentrant corner. It is assumed, that Ω has only one reentrant corner located at the origin. This is not a restriction since the difficulties introduced by such a corner are of local nature, such that a more general setting can be reduced to this case. We prove, that assumptions (A1)–(A10) of section 2 are fulfilled for several pairs of spaces (M_h, X_h). Consequently, the superconvergence result stated in Theorem 4.4 is valid in this case.

6.1 Regularity

The regularity results are similar to those stated in the case of a three-dimensional polyhedral domain. This means that the solution $(v, p) \in X \times M$ of problem (1.2) satisfies

$$v \in V_\beta^{2,p}(\Omega)^2 \text{ and } p \in V_\beta^{1,p}(\Omega) \quad \forall \beta > 2 - \lambda - \frac{2}{p}$$

and the a priori estimate

$$\|v\|_{V^{2,p}_\beta(\Omega)^2} + \|p\|_{V^{1,p}_\beta(\Omega)} \le c\|u\|_{L^p(\Omega)} \tag{6.1}$$

holds. Further Remark 5.2, Corollary 5.3, Corollary 5.4 and Lemma 5.5 are also valid in this two-dimensional setting. Of course you have to substitute 3 by 2 where necessary.

6.2 Discretization

We introduce a family of graded triangulations $(\mathcal{T}_h)_{h>0}$ of the domain Ω such that assumption (A1) is fullfilled. With global mesh parameter h, grading parameter μ and distance r_T of the elements to the corner,

$$r_T = \inf_{(x_1,x_2)\in T} \sqrt{x_1^2 + x_2^2},$$

we assume that the elementsize $h_T := \operatorname{diam} T$ satisfies

$$h_T \sim \begin{cases} h^{1/\mu} & \text{for } r_T = 0, \\ h r_T^{1-\mu} & \text{for } 0 < r_T \le R, \\ h & \text{for } r_T > R. \end{cases} \tag{6.2}$$

with some constant R and $\mu < \lambda$. Such a mesh was already used in [8] for the optimal control of the Poisson equation in a domain with reentrant corner.

The assumptions (A9) and (A10) do actually not depend on the spaces M_h and X_h but only on the regularity of the solution and the underlying mesh. Therefore we consider them first. The regularity of the components of the solution of the Stokes equation is similar to that of the Poisson equation. In [8] Apel and Winkler considered the optimal control problem (1.1), (1.3) governed by the Poisson equation in a three-dimensional domain with edges and corners. They used an isotropic mesh grading in order to achieve an optimal convergence rate under reduced regularity. The assumptions (A9) and (A10) are proved in Corollary 4.4 and Lemma 4.5 of that paper. These proofs can be easily adapted for a two-dimensional domain with reentrant corner. A componentwise consideration of the solution of (1.1)–(1.3) proves then the assertions (A9) and (A10), see also Section 5.

In the following we give examples of pairs of spaces that satisfy the assumptions (A2)–(A8). An overview can be found e.g. in [20].

a) Bernardi-Raugel-Fortin element

$$X_h = \{v_h \in H^1_0(\Omega)^2 : v_h|_T \in \mathcal{P}_1^+ \ \forall T \in \mathcal{T}_h\}$$
$$M_h = \{q_h \in L^2_0(\Omega) : q_h|_T \in \mathcal{P}_0 \ \forall T \in \mathcal{T}_h\}$$

where $\mathcal{P}_1^+ = \mathcal{P}_1 \oplus \operatorname{span}\{n_1\lambda_2\lambda_3, n_2\lambda_3\lambda_1, n_3\lambda_1\lambda_2\}$

b) $(\mathcal{P}_2^c, \mathcal{P}_0)$

c) Mini-element

$$X_h = \{v_h \in H_0^1(\Omega)^2 : v_h|_T \in \mathcal{P}_1^+ \ \forall T \in \mathcal{T}_h\}$$
$$M_h = \{q_h \in C(\bar{\Omega}) \cap L_0^2(\Omega) : q_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h\}$$

where $\mathcal{P}_1^+ = \mathcal{P}_1 \oplus \operatorname{span}\{\lambda_1\lambda_2\lambda_3\}$

d) Taylor-Hood element $(\mathcal{P}_2^c, \mathcal{P}_1^c)$

The spaces $V_\beta^{k,2}(\Omega)$ play the role of the spaces $H_\omega^k(\Omega)$. So Assumption (A2) follows from estimate (6.1).

Since $X_h \subset H_0^1(\Omega)^2$ for the elements given in a)–d) the Poincaré inequality stated in (A3) is trivially satisfied.

The assumption (A4) can be concluded from the embedding

$$V_{1-\mu}^{2,2}(\Omega) \hookrightarrow V_0^{1+\mu}(\Omega) \hookrightarrow W^{1+\mu,2}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

The first inclusion is proved in [36], the second one follows directly from the definition of the spaces and the last embedding is a conclusion from the Sobolev embedding theorem.

In order to proof (A5) we use the standard Lagrange interpolant. This is possible due to the fact that $(\mathcal{P}_h^1) \subset X_h$. Then the assumption (A5)(i) follows from Theorem 2 of [32], the proof of (A5)(iii) is similar. (A5)(ii) is a direct consequence of the embedding $H_\omega^2(\Omega) \hookrightarrow L^\infty(\Omega)$.

In the following we prove assumption (A6).

Proof of assumption (A6). For $\varphi_h \in X_h$ one has

$$\|\varphi_h\|_{L^\infty(T)} = \|\hat{\varphi}_h\|_{L^\infty(\hat{T})} \le c\|\hat{\varphi}_h\|_{L^p(\hat{T})} = c|T|^{-1/p}\|\varphi_h\|_{L^p(T)}$$

We choose the smallest element size $h_T = h^{1/\mu}$ and $p \ge \frac{2}{\mu}$. This yields

$$\|\varphi_h\|_{L^\infty(T)} \le ch^{-\frac{2}{\mu p}}\|\varphi_h\|_{L^p(T)} \le ch^{-1}\|\varphi_h\|_{L^p(T)}$$

what proves the assertion of (A6). □

For the conforming discretizations a)-d) the consistency error estimate (A7) is trivially satisfied.

For the proof of the inf-sup-condition for the above element pairs we refer to [20]

In summary we have shown that the assumptions (A1)–(A10) are fulfilled for a couple of conforming finite element discretization of the optimal control problem (1.1)–(1.3) in a polygonal domain with reentrant corner provided the mesh is sufficiently graded. Therefore the superconvergence result stated in Theorem 4.4 holds in this situation.

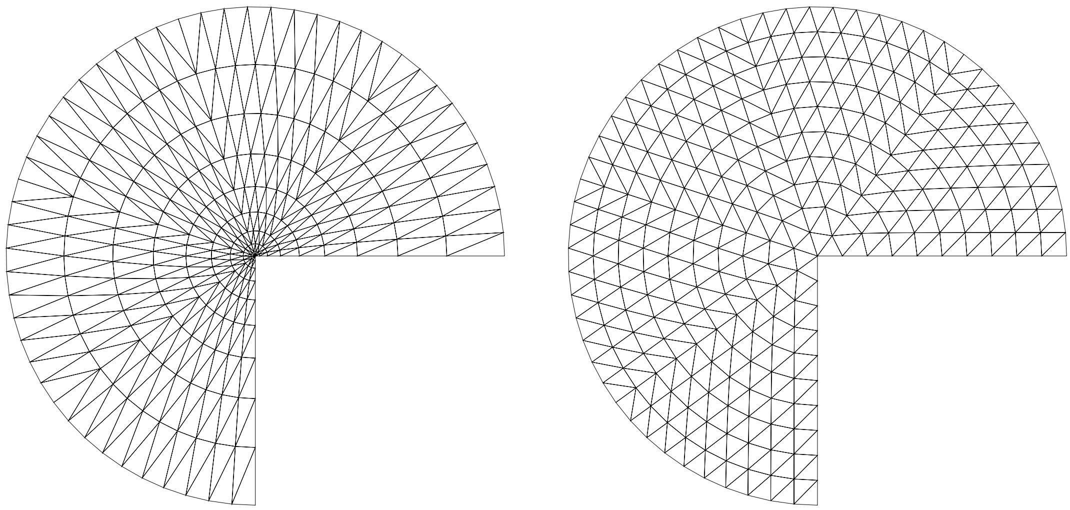

Figure 2: Graded mesh with $\mu = 0.4$ (left) and uniform mesh ($\mu = 1.0$)

Remark 6.1. *The assumptions (A1)–(A10) can be proved in a very similar way for three-dimensional domains with corner- and edge singularities and appropriate isotropic mesh grading (comp. [8]). On such isotropic meshes one can also use several well-known conforming element pairs like Bernardi-Raugel-Fortin element, Mini-element and Taylor-Hood element. A detailed overview can be found e.g. in [20].*

6.3 Numerical tests

In this subsection we consider the same optimal control problem as in Subsection 5.4, but now in the two-dimensional domain

$$\Omega = \left\{ (r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2 : 0 < r < 1, 0 < \varphi < \frac{3}{2}\pi \right\}.$$

The functions f, g and v_d are chosen such that

$$\bar{v} = \bar{w} = \begin{bmatrix} r^\lambda \Phi_1(\varphi) \\ r^\lambda \Phi_2(\varphi) \end{bmatrix}, \quad \bar{p} = -\bar{r} = r^{\lambda-1}\Phi_p(\varphi), \quad \bar{u} = \Pi_{[-1.0,0.1]}\left(-\frac{1}{\nu}\bar{w}\right)$$

is the exact solution of the optimal control problem. The functions Φ_1, Φ_2 and Φ_p are defined in Subsection 5.4 and $\lambda \approx 0.5445$ is again the smallest positive solution of (5.1). In our test we use the $(\mathcal{P}_2^c, \mathcal{P}_0)$ element.

In table 2 one can find the results for this case. For the appropriately graded mesh ($\mu = 0.4$, comp. Fig. 2) the predicted convergence rate of 2 can be seen. In case of a uniform mesh ($\mu = 1.0$) the convergence rate is slightly better than the theoretical value of $2\lambda \approx 1.09$.

	$\mu = 0.4$		$\mu = 1$	
ndof	value	rate	value	rate
697	1.05E-02		9.09E-03	
2642	2.99E-03	1.88	3.25E-03	1.54
10282	8.02E-04	1.94	1.16E-03	1.52
40562	2.08E-04	1.96	4.16E-04	1.49
161122	5.32E-05	1.98	1.52E-04	1.46
392377	2.20E-05	1.99	8.08E-05	1.42
642242	1.35E-05	1.99	5.73E-05	1.39
2564482	3.38E-06	1.99	2.25E-05	1.35

Table 2: L^2-error of the computed control $\tilde{u}_h$ on an isotropic, two-dimensional mesh

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